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WITH THE

CONSTRUCTION AND USE

OF

TABLES OF LOGARITHMS,

BOTH OF NUMBERS, AND FOR ANGLES.

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By J. C. SNOWBALL, M.A.

FELLOW OF ST JOHN'S COLLEGE, CAMBRIDGE.

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PLANE TRIGONOMETRY.

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The student is recommended to confine his attention at first to the Articles,

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PLANE TRIGONOMETRY.

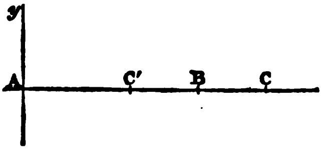
CHAPTER I.

ON THE METHOD OF REPRESENTING LINES AND ANGLES, AND ON THE DIVISIONS OF ANGLES.

1. *THE magnitudes of lines may be represented by algebraical quantities.*

The length of a line is measured by finding how many times it contains a fixed and definite line, as an inch or a yard, which has been previously fixed on as the unit of length. Thus, if an inch be taken as the unit of length, a line is said to be 30, if it contain an inch thirty times; and in like manner a line is called a, if it contain the unit of length a times.

2. *If lines drawn in a given direction from a fixed line be represented by positive quantities, then those which are drawn from it in a contrary direction will be represented by negative quantities.*

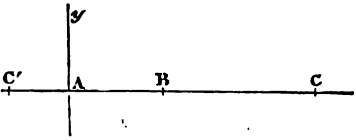


If to a line AB measured from A towards the *right* of the fixed line Ay it be required to add a given line, it is evident that AB must be produced to a point C such that BC shall be equal to that given line; and if from AB a given line has to be taken, a part BC' must be cut off from BA equal to the given line, and AC' will be the required remainder.

Let AB be represented by a, and BC by b, and make $BC' = BC$; then $AC' + C'B = AB$;

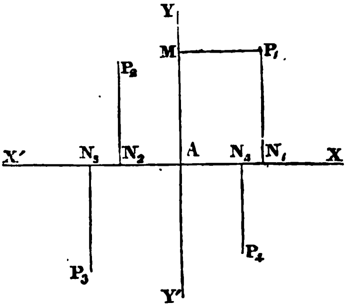
$$\therefore AC' = AB - C'B = a - b.$$

Now if b be greater than a, C' lies on the other side of A. And $AC' = a - b = -(b - a)$, which is a negative quantity equal in magnitude to the difference of the lines b and a; which difference, it is seen, lies in this case to the left of Ay.



Hence, if a line be represented by a negative quantity, it is meant that it is measured from Ay in a direction contrary to that in which the lines represented by positive quantities were measured.

3. Let XAX', YAY' be two fixed lines at right angles to each other, and produced, if necessary, indefinitely. Then the position of any point P_1 in the plane of these lines is known, if the magnitudes of the perpendiculars be known which are let fall from P_1 upon the lines XAX' and YAY', viz. P_1N_1 and P_1M (or AN_1, which is equal to P_1M).



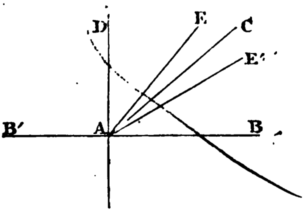
AN_1, N_1P_1, are called "*the co-ordinates of* P_1," and are said to be "*referred to the rectangular axes* AX *and* AY."

With respect to lines measured from A in the direction of the axis XAX', it is usual to call those *positive* which are measured to the *right* of the axis YAY', and therefore those will be *negative* which are measured to the *left* of YAY'.

With respect to lines measured in the direction of YAY', it is usual to call those *positive* which lie on the *upper* side of XAX', and therefore those will be *negative* which lie on the *lower* side of XAX'.

Thus if P_1 be a point in the space contained by AX and AY, its ordinate AN_1 which is measured in the direction of XAX' is *positive*, because it lies to the right of YAY'; and its ordinate N_1P_1 measured in the direction of YAY' is *positive*, because it lies on the upper side of XAX'. But if P_2 be the point, its ordinate AN_2 measured in the direction of XAX' is *negative*, because it lies to the left of YAY'; and the ordinate N_2P_2 measured in the direction of YAY' is *positive*, because it lies on the upper side of XAX'.

Similarly, the ordinates of P_3 and P_4 measured in the direction of XAX' are *negative* and *positive* respectively; and the ordinates measured in the direction of YAY' are *negative* in both cases.



4. **Def.** If a straight line revolve in one plane round its extremity A from a given position, as AB, into any other position, as AC, the inclination of AC to AB is called *an angle* $(\angle)$; and the angle is signified by the letters BAC or CAB, the middle letter being that placed at the point in which the two lines meet.

By continuing this revolving motion, the angle may be supposed to become of any magnitude whatever.

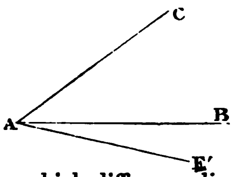
5. **Def.** If AD be equally inclined to the parts AB, AB' of the straight line BB', each of the angles BAD, $B'AD$ is called a *right* angle.

6. **Def.** An *acute* angle is less, and an *obtuse* angle is greater, than a *right* angle.

7. *If the angles formed by* AC *revolving in one direction,* (*as* BCD) *from the fixed line* AB, *be considered positive, then if* AC *revolve in the contrary direction from* AB, *it will trace out negative angles.*

If to the angle BAC, Fig. Art. 4, it be required to add a given angle, CA must move in the direction BCD through an angle CAE equal to the given angle, and the whole BAE will be the angle required. And if it be required to take a given angle from BAC, CA must evidently move in the contrary direction till it come into a position $E'A$, such that $\angle CAE'$ is equal to the angle to be subtracted.

$$\text{Then } \angle CAE' + \angle E'AB = \angle BAC;$$
$$\therefore \angle E'AB = \angle BAC - \angle CAE'.$$



Now if $\angle CAE'$ be greater than $\angle CAB$, $E'A$ lies on the other side of AB,

$$\text{and } \angle E'AB = \angle BAC - \angle CAE'$$
$$= -(\angle CAE' - \angle BAC),$$

a negative quantity, whose magnitude is the difference between the angles CAE' and BAC; which difference lies in this case on the lower side of AB.

Hence, when an angle is called *negative*, it is meant that *it is formed by the revolving line moving from the fixed line in a direction contrary to that in which it revolved to trace out positive angles.*

8. A right angle is divided by the English into 90 equal angles which are called degrees; a degree is subdivided into 60 minutes, and a minute into 60 seconds. And the magnitude of an angle is expressed by stating how many degrees and subdivisions of a degree are contained in the angle. If great accuracy be required, the parts of an angle which are less than a second are expressed in decimal parts of a second.

A degree and its subdivisions are thus indicated, $24^{0}, 50', 34''\cdot 7$, which denotes an angle containing 24 degrees, 50 minutes, 34 seconds, and seven tenths of a second.

9. By the French and other Continental Mathematicians, a right angle is divided into 100 equal angles called grades, a grade into 100 minutes, and a minute into 100 seconds; and the divisions are thus marked, $26^{g}, 24^{\backprime}, 32^{\backprime\backprime}\cdot 47$.

Now since $1^{\backprime} = \frac{1^{g}}{100} = \cdot 01^{g}$, and $1^{\backprime\backprime} = \frac{1^{\backprime}}{100} = \frac{1^{g}}{1000} = \cdot 0001^{g}$;

The above angle might have been written thus, $26^{g}\cdot 243247$.

Whence it appears that if the French division be adopted, arithmetical operations can be performed on angles in the same manner as on any other decimal fractions; an advantage which does not attend the English division of the angle.

10. *To find the relation between* E *and* F, *the number of Degrees and Grades contained in* the same *angle* BAC (Art. 14. Fig. 1.)

In the English division, $\frac{E^{0}}{90^{0}} = \frac{\text{given} \angle BAC}{\text{right} \angle}$;

In the French division, $\frac{F^{g}}{100^{g}} = \frac{\text{given} \angle BAC}{\text{right} \angle}$;

$$\therefore \frac{E}{90} = \frac{F}{100}; \quad \text{and} \quad \begin{cases} E = \frac{9}{10} F = F - \frac{F}{10}; \ldots (1). \\ F = \frac{10}{9} E = E + \frac{E}{9} \ldots (2). \end{cases}$$

NOTE. In making use of the latter of these formulæ it will be necessary to express the minutes and seconds of the angle in decimal parts of a degree, for E represents the number of *degrees* in the angle.

Ex. 1. To find how many degrees and minutes are contained in the angle $42^g,\ 34^{\backprime},\ 56^{\backprime\backprime}$.

$$\begin{array}{r}
F=42\cdot3456\\
\dfrac{F}{10}=4\cdot23456\\
\hline
\therefore\ E=F-\dfrac{F}{10}=38\cdot11104\\
60\\
\hline
6\cdot6624\\
60\\
\hline
39\cdot744\\
\hline
\end{array}$$

$\therefore\ E=38^\circ,\ 6',\ 39''\cdot74$, retaining the tenths and hundredths and neglecting the *thousandth* parts of a second.

Ex. 2. Find how many grades and minutes are contained in the angle $24^\circ,\ 51,\ 45''$.

First reducing the minutes and seconds to the decimal parts of a degree,

$$\begin{array}{r}
60\,)\,45''\\
\hline
60\,)\,51'\cdot75\\
\hline
.8625\,;\\
\hline
\end{array}
\qquad\qquad
\begin{array}{r}
\therefore\ E=24\cdot8625\\
\dfrac{E}{9}=2\cdot7625\\
\hline
\therefore\ E+\dfrac{E}{9}=27\cdot6250
\end{array}$$

$$\text{and}\ \therefore\ F=27^g,\ 62^{\backprime},\ 50^{\backprime\backprime}.$$

11. **Def.** *The Complement of an angle is its* defect *from a right angle.*

Thus, $90^\circ-\ 24^\circ,\ 32'=65^\circ,\ 28'$ is the complement of $24^\circ,\ 32'$.
$90^\circ-110^\circ,\ 15'=-(20^\circ,\ 15')$ is the complement of $110^\circ,\ 15'$.

12. **Def.** *The Supplement of an angle is its* defect *from two right angles.*

Thus, $180^\circ-\ 56^\circ,\ 20'=123^\circ,\ 40'$ is the supplement of $56^\circ,\ 20'$.
$180^\circ-186^\circ,\ 12'=-(6^\circ,\ 12')$ is the supplement of $186^\circ,\ 12'$.

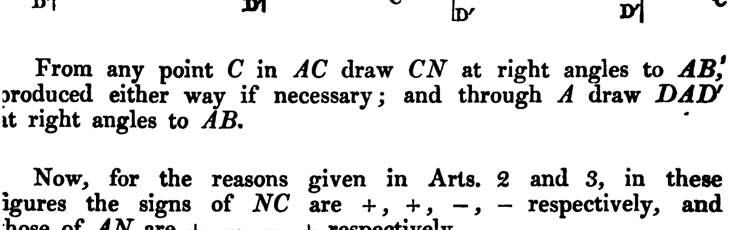
A few examples to each chapter are placed after the third Appendix.

CHAPTER II.

THE GONIOMETRICAL RATIOS, AND SOME FORMULÆ CONNECTING THEM WITH EACH OTHER.

13. **Def.** *Plane Trigonometry*, in its original meaning, implies *the measuring of plane triangles;* in its extended signification, it treats of the formulæ connecting the relations of angles with each other, and of the determination of the parts of plane. rectilineal figures from sufficient data.

14. Let a straight line revolve from the fixed line AB round the point A, in the direction of the letters B, D, B', D', and let there be A degrees in the angle BAC.



From any point C in AC draw CN at right angles to AB, produced either way if necessary; and through A draw DAD' at right angles to AB.

Now, for the reasons given in Arts. 2 and 3, in these igures the signs of NC are $+$, $+$, $-$, $-$ respectively, and those of AN are $+$, $-$, $-$, $+$ respectively.

15. **Definitions.** See the figures of the last Article.

1. $\frac{NC}{AC}$ is the *Sine* of the $\angle BAC$; or, Sin $\angle BAC = \frac{NC}{AC}$;

2. $\frac{AN}{AC}$ is the *Cosine* of the $\angle BAC$; or, Cos $\angle BAC = \frac{AN}{AC}$.

3. $\frac{NC}{AN}$ is the *Tangent* of the $\angle BAC$; or, Tan $\angle BAC = \frac{NC}{AN}$.

4. $\frac{AC}{AN}$ is the *Secant* of the $\angle BAC$; or, Sec $\angle BAC = \frac{AC}{AN}$.

5. $1 - \cos \angle BAC$ is the *Versed-sine* of the $\angle BAC$;

or, Versin $\angle BAC = 1 - \cos \angle BAC$.

6. The tangent of the complement of the $\angle BAC$ is called the *Cotangent* of the $\angle BAC$;

or, Cot $\angle BAC = \tan (90^\circ - \angle BAC)$.

COR. If $90^\circ - \angle BAC$ be the original angle, its complement is $\angle BAC$, Art. 11,

$\therefore$ Cotan $(90^\circ - \angle BAC) = \tan \angle BAC$,

or, Tan $\angle BAC = \text{cotan} (90^\circ - \angle BAC)$.

7. The Secant of the Complement of the $\angle BAC$ is called the *Cosecant* of the $\angle BAC$;

or, Cosec $\angle BAC = \sec (90^\circ - \angle BAC)$.

COR. If $90^\circ - \angle BAC$ be the original angle, its complement is $\angle BAC$, Art. 11,

$\therefore$ Cosec $(90^\circ - \angle BAC) = \sec \angle BAC$;

or, Sec $\angle BAC = \text{cosec} (90^\circ - \angle BAC)$.

16. The Cosine of $\angle BAC$ might have been defined to be the sine of the Complement of $\angle BAC$.

For Cos $\angle BAC = \frac{AN}{AC} = \sin \angle ACN = \sin (90^\circ - \angle BAC)$. Fig. 1.

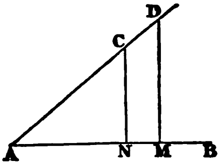
Also, Sin $\angle BAC = \frac{NC}{AC} = \cos \angle ACN = \cos (90^\circ - \angle BAC)$,

or the Sine of an angle is equal to the Cosine of its Complement.

For the sake of convenience an angle will generally hereafter be indicated by a single letter, as Sin A, Cos A, Tan B.

17. So long as the magnitude of the angle is unaltered, its Sine, Cosine, Tangent, &c. remain the same, whatever be the magnitude of AC.

Let D be any other point in AC, DM perpendicular to AB.



Then by definition, Sin $A = \frac{NC}{AC}$; or Sin $A = \frac{MD}{AD}$.

But by similar triangles $\frac{NC}{AC} = \frac{MD}{AD}$, or Sin A is the same wherever in the line AC the point C be situated. Similarly it may be shewn that Cos A, Tan A, Sec A ... are invariable quantities so long as the magnitude of A remains unaltered.

Hence, if any of the quantities Sin A, Cos A, Tan A, Sec A... be given, the angle A may be determined.

18. Def. The Ratios which are called the Sine, Cosine, Tangent, &c., of any angle are termed "The Goniometrical Ratios," because when any one of them is given the angle may be determined to which it belongs.

19. *To express* Versin A, Cot A, Cosec A, *in terms of the sides of the triangle* ANC. (Fig 1. Art. 20.)

(1) $\text{Versin } A = 1 - \cos A = 1 - \frac{AN}{AC}$.

(2) $$\text{Cot } A = \tan (90^\circ - A) = \tan ACN$$

$$= \frac{NA}{CN}, \text{ by def. of the } \textit{tangent}.$$

(3) $$\text{Cosec } A = \sec (90^\circ - A) = \sec ACN$$

$$= \frac{CA}{CN}, \text{ by def. of the } \textit{secant}.$$

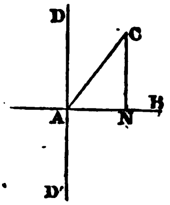
20. *To trace the variation in the* signs *of* Sin A, Cos A, Tan A, Sec A, *as* A *increases from* 0° *to* 360°.

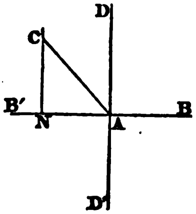
(1) Sin $A=\frac{NC}{AC}$, and therefore has the same sign in any case as NC has; for AC, which lies in the direction of neither of the lines AB and AD, cannot change its sign, and is always to be reckoned as positive.

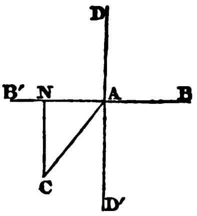
Hence (14) Sin A is positive if A be between 0° and 180°, (figs. 1, 2.); and is negative if A be between 180° and 360°. (Figs. 3, 4.)

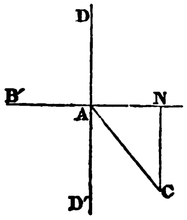
(2) Cos $A=\frac{AN}{AC}$, and therefore has the same sign as AN.

Hence (14) Cos A is positive if A be between 0° and 90°, or between 270° and 360° (figs. 1, 4.); and is negative if A be between 90° and 270°. (Figs. 2, 3.)









(3) Tan $A=\frac{NC}{AN}$, and is therefore positive or negative according as NC and AN have the same or different signs.

Hence (14) Tan A is positive if A be between 0° and 90°, or between 180° and 270°, (figs. 1, 3.); and it is negative if A be between 90° and 180°, or between 270° and 360°. (Figs. 2, 4.)

(4) Sec $A=\frac{AC}{AN}$, and therefore has the same sign as AN.

Hence Sec A is positive if A be between 0° and 90°, or between 270° and 360°; and it is negative if A be between 90° and 270°.

21. *To trace the variations in the* magnitudes *of the Sine, Cosine, Tangent, and Secant, as the angle increases from* 0° *to* 360°. (Figs. Art. 20.)

Since (17) the values of the Sine, Cosine, Tangent, and Secant are not affected by the magnitude of AC, suppose this line to remain of the same magnitude while the angle A increases from 0° to 360°.

Now (fig. 1.) as AC revolves from the position AB into the position AD, NC increases in magnitude from 0 to AC, and is positive; and AN decreases from AC to 0, and is positive.

As (fig. 2.) AC revolves from the position AD into the position AB', NC decreases in magnitude from AC to 0, and is positive; and AN increases from 0 to AC, and is negative.

As (fig. 3.) AC revolves from AB' to AD', NC increases in magnitude from 0 to AC, and is negative; and AN decreases from AC to 0, and is negative.

As (fig. 4.) AC revolves from AD' to AB, NC decreases in magnitude from AC to 0, and is negative; and AN increases from 0 to AC, and is positive.

Hence it appears, that as

A changes from	0° to 90°	90° to 180°	180° to 270°	270° to 360°
Sin A $\left(\frac{NC}{AC}\right)$...	$\frac{0}{AC}$ to $\frac{+AC}{AC}$	$\frac{+AC}{AC}$ to $\frac{0}{AC}$	$\frac{0}{AC}$ to $\frac{-AC}{AC}$	$\frac{-AC}{AC}$ to $\frac{0}{AC}$
Cos A $\left(\frac{AN}{AC}\right)$...	$\frac{+AC}{AC} \cdots \frac{0}{AC}$	$\frac{0}{AC} \cdots \frac{-AC}{AC}$	$\frac{-AC}{AC} \cdots \frac{0}{AC}$	$\frac{0}{AC} \cdots \frac{+AC}{AC}$
Tan A $\left(\frac{NC}{AN}\right)$...	$\frac{0}{+AC} \cdots \frac{+AC}{0}$	$\frac{+AC}{0} \cdots \frac{0}{-AC}$	$\frac{0}{-AC} \cdots \frac{-AC}{0}$	$\frac{-AC}{0} \cdots \frac{0}{+AC}$
Sec A $\left(\frac{AC}{AN}\right)$...	$\frac{AC}{+AC} \cdots \frac{AC}{0}$	$\frac{AC}{0} \cdots \frac{AC}{-AC}$	$\frac{AC}{-AC} \cdots \frac{AC}{0}$	$\frac{AC}{0} \cdots \frac{AC}{+AC}$

These changes in sign and magnitude of the Sine, Cosine, Tangent, and Secant may be thus exhibited; the signs which belong to them in each right angle being written in a bracket. The symbol ∞ indicates an infinitely large quantity.

A being between }.	0° and 90°	90° and 180°	180° and 270°	270° and 360°
Sin A......	0 and 1, (+)	1 and 0, (+)	0 and −1, (−)	−1 and 0, (−)
Cos A......	1 ... 0, (+)	0 ... −1, (−)	−1 ...... 0, (−)	0...... 1, (+)
Tan A......	0 ... ∞, (+)	∞ .... 0, (−)	0 ...... ∞, (+)	∞...... 0, (−)
Sec A......	1 ... ∞, (+)	∞ ...−1, (−)	−1 ...... ∞, (−)	∞...... 1, (+)

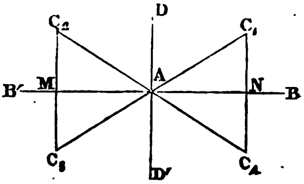
Since Cos A is never greater than unity, Versin A (or $1-\cos A$) is always positive; and its greatest value is when A becomes 180°, when Cos A becomes -1, and Versin A becomes 2.

22. *To shew that*

Sin A = sin (180° − A), *or* = − sin (180° + A), *or* = − sin (360° − A), *or* = − sin (− A).

Let $BAC_1 = A = B'AC_2 = B'AC_3 = BAC_4$;
and $AC_1 = AC_2 = AC_3 = AC_4$.

Join C_1C_4, and C_2C_3.



It may easily be shewn that the angles at M and N are right angles; and that NC_1, MC_2, MC_3, NC_4 are equal in magnitude,—as are also AN and AM.

$$\text{Now, } \operatorname{Sin} A = \frac{NC_1}{AC_1} = \frac{MC_2}{AC_2} = \sin BAC_2$$

$$= \sin (BAD + DAB' - B'AC_2)$$

$$= \sin (180^\circ - A) \quad \ldots\ldots\ldots\ldots (1).$$

$$\text{Again, } \operatorname{Sin} A = \frac{NC_1}{AC_1} = \frac{-MC_3}{AC_3}, \text{ since } NC_1 = -MC_3,$$

$$= -\frac{MC_3}{AC_3}$$

$$= -\sin (BAD + DAB' + B'AC_3)$$

$$= -\sin (180^\circ + A) \quad \ldots\ldots\ldots\ldots\ldots (2).$$

$$\text{Again, } \operatorname{Sin} A = \frac{NC_1}{AC_1} = \frac{-NC_4}{AC_4} = -\frac{NC_4}{AC_4};$$

but $\frac{NC_4}{AC_4}$ is either the sine of the *positive* angle

$$\{BAD + DAB' + B'AD' + (D'AB - BAC_4)\},$$

or the sine of the *negative* angle BAC_4, (7);

$$\therefore \operatorname{Sin} A = -\sin (360^\circ - A) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\text{or } = -\sin (-A) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

23. In like manner it might be shewn that

(1) $\text{Cos}\,A = -\cos(180^\circ - A), = -\cos(180^\circ + A), = \cos(360^\circ - A), = \cos(-A)$.

(2) $\text{Tan}\,A = -\tan(180^\circ - A), = \tan(180^\circ + A), = -\tan(360^\circ - A), = -\tan(-A)$.

(3) $\text{Sec}\,A = -\sec(180^\circ - A), = -\sec(180^\circ + A), = \sec(360^\circ - A), = \sec(-A)$.

24. If any angle, as BAC_1, be increased by 360°, the line which bounds it will come into the same position again, and the sine of the angle will therefore remain unaltered. Wherefore $\sin A$ is in all cases the same with $\sin(360^\circ + A)$; and in like manner, $\sin(360^\circ + A) = \sin(2 \times 360^\circ + A)$, and so on. If, therefore, n be *any* positive integer,

$$\text{Sin}\,A = \sin(n \,.\, 360^\circ + A) = \sin(2n \,.\, 180^\circ + A) \ldots\ldots\ldots (1).$$

Similarly, $\text{Sin}\,A = \sin(180^\circ - A)$ Art. 22. (1),

$$= \sin\{2n \,.\, 180^\circ + (180^\circ - A)\}$$

$$= \sin\{(2n+1)\, 180^\circ - A\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

In like manner it appears from (2) and (4) of Art. 22, that

$$\text{Sin}\,A = -\sin\{(2n+1)\, 180^\circ + A\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$\text{Sin}\,A = -\sin(2n \,.\, 180^\circ - A) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

25. By the same process of reasoning it may be proved from the formulæ of (23) that

$$\text{Cos}\,A = \cos(2n \,.\, 180^\circ + A), \qquad \text{or} = -\cos\{(2n+1)\, 180^\circ - A\},$$

$$\text{or} = -\cos\{(2n+1)\, 180^\circ + A\}, \ \text{or} = \cos(2n \,.\, 180^\circ - A).$$

And, $\text{Tan}\,A = \tan(2n \,.\, 180^\circ + A)$, $\qquad$ or $= -\tan\{(2n+1) \,.\, 180^\circ - A\}$,

$$\text{or} = \tan\{(2n+1) \,.\, 180^\circ + A\}, \ \text{or} = -\tan(2n \,.\, 180^\circ - A).$$

Similarly it might be proved that,

$$\operatorname{Sec} A = \sec(2n\,.\,180^\circ + A), \qquad \text{or} = -\sec\{(2n+1)\,.\,180^\circ - A\},$$

$$\text{or} = -\sec\{(2n+1)\,.\,180^\circ + A\}, \quad \text{or} = \sec(2n\,.\,180^\circ - A)^*.$$

26. From Arts. 16, 22, 23, it will appear that

$$\begin{array}{ll} \operatorname{Sin} A = \cos(90^\circ - A) & \operatorname{Cos} A = \sin(90^\circ - A) \\ \operatorname{Sin} A = \sin(180^\circ - A). & \operatorname{Cos} A = -\cos(180^\circ - A). \\ \operatorname{Tan} A = \cot(90^\circ - A) & \operatorname{Sec} A = \operatorname{cosec}(90^\circ - A) \\ \operatorname{Tan} A = -\tan(180^\circ - A). & \operatorname{Sec} A = -\sec(180^\circ - A). \end{array}$$

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[*] In (24) it has been shewn, BAC being an angle of A degrees, that

$$\operatorname{Sin} A = \sin(2n\,.\,180^\circ + A), \quad \text{or} = \sin\{(2n+1)\,.\,180^\circ - A\}.$$

Also, as the negative angles $BAD' + D'AB' + B'AC_2$ {or $-(180^\circ + A)$}, and $BAD' + D'AB' + B'AD + (DAB - BAC_1)$ {or $-(360^\circ - A)$}, have the same sine as A, and since also if either of these angles be increased by the negative angle -360° taken any number of times its sine remains the same, therefore

$$\operatorname{Sin} A = \sin\{-n\,.\,360^\circ - (180^\circ + A)\}, \quad \text{or} = \sin\{-n\,.\,360^\circ - (360^\circ - A)\},$$

$$\therefore \; = \sin\{-(2n+1)\,.\,180^\circ - A\}, \quad \text{or} = \sin\{-2(n-1)\,.\,180^\circ + A\},$$

(where the latter angle is of the *form*, $-2n\,.\,180^\circ + A$; n being some integer).

It appears, then, that A being the least angle whose sine is $\frac{NC_1}{AC_1}$, all angles have the same sine as A, which are comprehended under the formulæ $2n\,.\,180^\circ + A$ and $(2n+1)\,180^\circ - A$, where n may be 0 or any integer, negative as well as positive.

In the same manner, since all angles which are bounded by the lines AB and AC_1, and by AB and AC_4, have the same cosine, $\left(\frac{AN}{AC_1}\right)$, it may be proved that all angles whose cosine is equal to this are comprehended under the formula

$$2n\,.\,180^\circ \pm A;$$

And all angles which have the same tangent as A, (being those bounded by AB and either of the lines AC_1 and AC_3), are comprehended under the formula

$$n\,.\,180^\circ + A:$$

And n, in all these cases, may be 0, or any integer, negative as well as positive.

That is,

The Sine of an angle = cosine of its complement,
or, = sine of its supplement.
Cosine of an angle = sine of its complement,
or, = − cosine of its supplement.
Tangent of an angle = cotangent of its complement,
or, = − tangent of its supplement.
Secant of an angle = cosecant of its complement,
or, = − secant of its supplement.

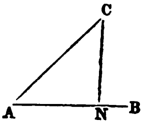
Note. These relations between the sine, cosine, tangent, and secant, of an angle and the sine, cosine, tangent, and secant of its complement and supplement, are perpetually occurring in practice, and will often be made use of in the following pages.

These two formulæ also are often useful;

$$\text{Sin}\,A = \cos(90^\circ - A) = -\cos\{180^\circ - (90^\circ - A)\} = -\cos(90^\circ + A);$$

$$\text{Cos}\,A = \sin(90^\circ - A) = \sin\{180^\circ - (90^\circ - A)\} = \sin(90^\circ + A).$$

27. It will be found necessary to carry in memory the following expressions,



$$(1)\quad \text{Tan}\,A = \frac{NC}{AN} = \frac{\frac{NC}{AC}}{\frac{AN}{AC}} = \frac{\sin A}{\cos A}.$$

$$(2)\quad \text{Sec}\,A = \frac{AC}{AN} = \frac{1}{\frac{AN}{AC}} = \frac{1}{\cos A};\quad \therefore\ \text{Cos}\,A = \frac{1}{\sec A}.$$

$$(3)\quad \text{Cot}\,A = \frac{AN}{NC} = \frac{\frac{AN}{AC}}{\frac{NC}{AC}} = \frac{\cos A}{\sin A}.$$

(4) $\text{Cot}\,A = \dfrac{AN}{NC} = \dfrac{1}{\dfrac{NC}{AN}} = \dfrac{1}{\tan A}$;

$$\therefore\ \text{Tan}\,A = \frac{1}{\cot A}.$$

(5) $\text{Cosec}\,A = \dfrac{AC}{NC} = \dfrac{1}{\dfrac{NC}{AC}} = \dfrac{1}{\sin A}$; $\quad \therefore\ \text{Sin}\,A = \dfrac{1}{\text{cosec}\,A}$.

(6) $$AC^2 = NC^2 + AN^2;\ \therefore\ 1 = \left(\frac{NC}{AC}\right)^2 + \left(\frac{AN}{AC}\right)^2,$$

or $1 = (\sin A)^2 + (\cos A)^2$;*

$$\therefore \begin{cases} \text{Sin}\,A = \sqrt{(1 - \cos^2 A)}, \\ \text{Cos}\,A = \sqrt{(1 - \sin^2 A)}. \end{cases}$$

(7) $AC^2 = AN^2 + NC^2$;

$$\therefore\ \left(\frac{AC}{AN}\right)^2 = 1 + \left(\frac{NC}{AN}\right)^2,\ \text{ or } \text{Sec}^2 A = 1 + \tan^2 A;$$

$$\therefore \begin{cases} \text{Sec}\,A = \sqrt{(1 + \tan^2 A)}, \\ \text{Tan}\,A = \sqrt{(\sec^2 A - 1)}. \end{cases}$$

(8) $AC^2 = AN^2 + NC^2$;

$$\therefore\ \left(\frac{AC}{NC}\right)^2 = \left(\frac{AN}{NC}\right)^2 + 1;\ \text{ or } \text{Cosec}^2 A = \text{cotan}^2 A + 1;$$

$$\therefore \begin{cases} \text{Cosec}\,A = \sqrt{(1 + \cot^2 A)}, \\ \text{Cot}\,A = \sqrt{(\text{cosec}^2 A - 1)}. \end{cases}$$

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* The powers of the *goniometrical ratios*, as $(\sin A)^2$, $(\cos A)^3$, $(\tan A)^n$, are generally written thus, $\sin^2 A$, $\cos^3 A$, $\tan^n A$.

28. By means of the expressions proved in the last Article the value of any one of the quantities defined in (15) may be found in terms of any other of them. For example:

(1) $$\text{Tan}\,A = \frac{\sin A}{\sqrt{(1-\sin^2 A)}}.$$

For $\text{Tan}\,A = \frac{\sin A}{\cos A}$, Art. 27. (1); $= \frac{\sin A}{\sqrt{(1-\sin^2 A)}}$; (27. 6).

(2) $$\text{Tan}\,A = \frac{\sin A}{\cos A} = \frac{\sqrt{(1-\cos^2 A)}}{\cos A}.$$

(3) $$\text{Sin}\,A = \frac{\sin A}{\cos A}\,.\,\cos A = \tan A\,.\,\frac{1}{\sec A}$$

$$= \frac{\tan A}{\sec A} = \frac{\tan A}{\sqrt{(1+\tan^2 A)}}.$$

29. The formulæ proved in the last Article will often be found useful to the analyst. The same method of proof is applicable to all other questions of the same kind. Thus, required to express the cosine of an angle in terms of the cosecant, and the cosecant in terms of the versed-sine.

(1) $$\text{Cos}\,A = \sqrt{(1-\sin^2 A)} = \sqrt{\left\{1-\frac{1}{\text{cosec}^2 A}\right\}} = \frac{\sqrt{(\text{cosec}^2 A - 1)}}{\text{cosec}\,A}.$$

(2) $$\text{Cosec}\,A = \frac{1}{\sin A} = \frac{1}{\sqrt{(1-\cos^2 A)}}$$

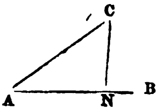
$$= \frac{1}{\sqrt{1-(1-\text{versin}\,A)^2}} = \frac{1}{\sqrt{(2\,\text{versin}\,A - \text{versin}^2 A)}}.$$

30. The following Table contains those values of Sin A, Cos A, Tan A and Sec A, which are most usefully remembered.

Sin A	Cos A	Tan A	Sec A
$\sqrt{(1-\cos^2 A)}$	$\sqrt{(1-\sin^2 A)}$	$\frac{\sin A}{\sqrt{(1-\sin^2 A)}}$	$\frac{1}{\sqrt{(1-\sin^2 A)}}$
$\frac{\tan A}{\sqrt{(1+\tan^2 A)}}$	$\frac{1}{\sqrt{(1+\tan^2 A)}}$	$\frac{\sqrt{(1-\cos^2 A)}}{\cos A}$	$\frac{1}{\cos A}$
$\frac{\sqrt{(\sec^2 A-1)}}{\sec A}$	$\frac{1}{\sec A}$	$\sqrt{(\sec^2 A-1)}$	$\sqrt{(1+\tan^2 A)}$

31. *If* A *be less than half a right angle, or* 45°, *Cos* A *is greater than Sin* A.

Let ∠ NAC be less than 45°.



Then, since ∠ NAC + ∠ $NCA = 90°$, ∠ NCA is greater than 45°. And in every triangle the greater side is opposite to the greater angle, (Eucl. I. 19.):

$$\therefore\ AN > NC; \qquad \therefore\ \frac{AN}{AC} > \frac{NC}{AC}, \qquad \text{or } \operatorname{Cos} A > \sin A.$$

Similarly, for angles between 45° and 90°, it may be shewn that the Cosine is *less* than the Sine.

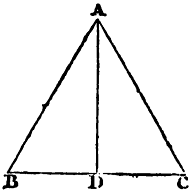
32. *To find the Sines, Cosines, and Tangents of* 45°, 30°, *and* 60°.

(1) (Fig. Art. 27.) Let $\angle NAC = 45°$; $\therefore \angle NCA = 90° - \angle NAC = 45°$;

$$\therefore\ \frac{AN}{AC} = \frac{NC}{AC}; \qquad \text{or } \operatorname{Sin} 45° = \operatorname{Cos} 45°.$$

$$\text{Also,}\quad AC^2 = AN^2 + CN^2 = 2\,AN^2;$$

$$\therefore\ \operatorname{Sin} 45° = \frac{AN}{AC} = \frac{1}{\sqrt{2}}; \qquad \operatorname{Cos} 45° = \frac{1}{\sqrt{2}}; \qquad \operatorname{Tan} 45° = \frac{NC}{AN} = 1.$$



(2) Let ABC be an equilateral and equiangular triangle; each of its angles, therefore, is $\frac{1}{3}$ of two right angles, or 60°.

Let AD be perpendicular to BC;

$$\therefore\ BD = DC = \tfrac{1}{2} BC = \tfrac{1}{2} AB;$$

$$\text{and } \angle BAD = \angle DAC = 30°;$$

$$\therefore\ \operatorname{Sin} 30° = \frac{DB}{AB} = \frac{\frac{1}{2} AB}{AB} = \frac{1}{2}.$$

$$\operatorname{Cos} 30° = \sqrt{\{1 - \sin^2 30\}} = \sqrt{(1 - \tfrac{1}{4})} = \frac{\sqrt{3}}{2},$$

$$\operatorname{Tan} 30° = \frac{\sin 30°}{\cos 30°} = \frac{1}{\sqrt{3}}.$$

(3) $\text{Sin } 60^0 = \cos (90^0 - 60^0), \ldots (16), \quad = \cos 30^0 = \frac{\sqrt{3}}{2},$

$\text{Cos } 60^0 = \sin (90^0 - 60^0) = \sin 30^0 = \frac{1}{2},$

$\text{Tan } 60^0 = \frac{\sin 60^0}{\cos 60^0} = \sqrt{3}.$

33. Equations like the following may often be solved by means of the relations established in (27) between the different Goniometrical Ratios.

Ex. 1. *From the equation,* $\text{Sin}^2 A + 5 \cos^2 A = 3$, *required the value of* Sin A.

Since $\text{Cos}^2 A = 1 - \sin^2 A$, the equation becomes $\text{Sin}^2 A + 5 . \{1 - \sin^2 A\} = 3$; whence $4 \sin^2 A = 2$; and $\text{Sin } A = \frac{1}{\sqrt{2}}$.

Ex. 2. *From the equations* $\text{Sin A} = \text{m} . \sin \text{B}$, *and* $\text{Tan A} = \text{n} . \tan \text{B}$, *required the values of* Sin A *and* Cos B.

For Sin A put x, and for Cos B put y; $\therefore \sin B = \sqrt{1 - y^2}$,

$$\text{Tan } A = \frac{\sin A}{\sqrt{\{1 - \sin^2 A\}}} = \frac{x}{\sqrt{1 - x^2}}, \quad (30),$$

$$\text{Tan } B = \frac{\sqrt{\{1 - \cos^2 B\}}}{\cos B} = \frac{\sqrt{1 - y^2}}{y}, \quad (30).$$

Making these substitutions, the equations become

$$x = m \sqrt{1 - y^2}; \quad \text{and } \frac{x}{\sqrt{1 - x^2}} = n . \frac{\sqrt{1 - y^2}}{y};$$

$$\text{whence, } x = \sin A = \sqrt{\frac{m^2 - n^2}{1 - n^2}}; \quad \text{and } y = \cos B = \frac{n}{m} \sqrt{\frac{1 - m^2}{1 - n^2}}.$$

Ex. 3. *Given* $\left\{ \begin{array}{l} \text{m} = \text{cosec A} - \sin \text{A} \\ \text{n} = \sec \text{A} - \cos \text{A} \end{array} \right\}$, *required to find an equation between* m *and* n *in which the angle* A *shall not appear.*

The equations severally give, $m = \frac{\cos^2 A}{\sin A}$, and $n = \frac{\sin^2 A}{\cos A}$;

$$\therefore \frac{n}{m} = \frac{\sin^3 A}{\cos^3 A} = \tan^3 A; \quad \therefore \tan A = \frac{n^{\frac{1}{3}}}{m^{\frac{1}{3}}}.$$

$$\text{Also, } m^2 = \frac{\cos^4 A}{\sin^2 A} = \frac{\cos^2 A}{\tan^2 A} = \frac{1}{\tan^2 A . \sec^2 A},$$

$$\therefore m^2 . \tan^2 A . (1 + \tan^2 A) = 1;$$

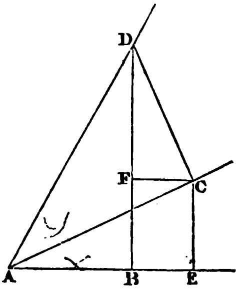
whence, by substituting its value for $\tan A$,

$$m^{\frac{2}{3}} . n^{\frac{2}{3}} . (m^{\frac{2}{3}} + n^{\frac{2}{3}}) = 1.$$

CHAPTER III.

GONIOMETRICAL FORMULÆ INVOLVING MORE THAN ONE ANGLE.

34. *Given the Sines and Cosines of two Angles, to find the Sines and Cosines of the angles equal to their Sum and to their Difference.*



Let *BAC, CAD* be two angles represented by *A* and *B* respectively.

From *D*, any point in *AD*, draw *DB* and *DC* perpendiculars on *AB* and *AC*; and from *C* draw *CE* and *CF* perpendiculars on *AB* and *DB*.

Then *FE* is a rectangle; $FB = CE$, and $FC = BE$.

$\angle CDF = 90^\circ - \angle DCF = \angle FCA$

$= A$, since *CF* is parallel to *AE*.

$$\text{Now, } \sin(A+B) = \frac{BD}{AD} = \frac{BF+FD}{AD} = \frac{EC}{AD} + \frac{FD}{AD}$$

$$= \frac{EC}{AC}\cdot\frac{AC}{AD} + \frac{FD}{DC}\cdot\frac{DC}{AD}$$

$$= \sin A \,.\, \cos B + \cos A \,.\, \sin B \ldots\ldots\ldots\ldots\ldots(1).$$

$$\text{Also, } \cos(A+B) = \frac{AB}{AD} = \frac{AE-EB}{AD} = \frac{AE}{AD} - \frac{FC}{AD}$$

$$= \frac{AE}{AC}\cdot\frac{AC}{AD} - \frac{FC}{CD}\cdot\frac{CD}{AD}$$

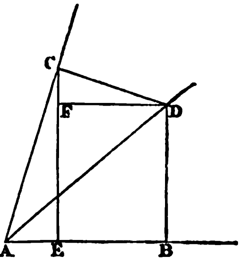
$$= \cos A \,.\, \cos B - \sin A \,.\, \sin B \ldots\ldots\ldots\ldots\ldots(2).$$

Again; let $\angle BAC = A$, and $\angle CAD = B$.

From D, any point in AD, draw DB and DC perpendiculars on AB and AC, CE a perpendicular from C on AB, DF perpendicular to CE.

FB is a rectangle; $FE = DB$, and $FD = EB$;

$$\angle DCF = 90^\circ - \angle ACE = A.$$



$$\text{Then, Sin } (A-B) = \frac{BD}{AD} = \frac{EC-CF}{AD} = \frac{EC}{AD} - \frac{CF}{AD}$$

$$= \frac{EC}{AC} \cdot \frac{AC}{AD} - \frac{CF}{CD} \cdot \frac{CD}{AD}$$

$$= \sin A . \cos B - \cos A . \sin B \ldots\ldots\ldots\ldots (3).$$

$$\text{Also, } \quad \text{Cos } (A-B) = \frac{AB}{AD} = \frac{AE+EB}{AD} = \frac{AE}{AD} + \frac{FD}{AD}$$

$$= \frac{AE}{AC} \cdot \frac{AC}{AD} + \frac{FD}{CD} \cdot \frac{CD}{AD}$$

$$= \cos A . \cos B + \sin A . \sin B \ldots\ldots\ldots\ldots (4).$$

Ex. Having given the Sines and Cosines of 45° and 30°, required the Sine and Cosine of 75°, and of 15°.

$$\text{Sin } 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}; \ \sin 30^\circ = \tfrac{1}{2}, \ \cos 30^\circ = \frac{\sqrt{3}}{2}, \ (32).$$

$$\text{Sin } 75^\circ = \sin (45^\circ + 30^\circ) = \sin 45^\circ . \cos 30^\circ + \cos 45^\circ . \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{1}{2\sqrt{2}} \cdot (\sqrt{3}+1).$$

$$\text{Similarly, } \quad \text{Cos } 75^\circ = \cos (45^\circ + 30^\circ) = \frac{1}{2\sqrt{2}} \cdot (\sqrt{3}-1).$$

$$\text{Sin } 15^\circ = \sin (45^\circ - 30^\circ) = \frac{1}{2\sqrt{2}} \cdot (\sqrt{3}-1).$$

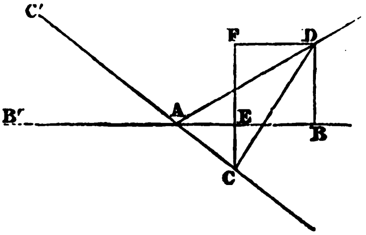
$$\text{Cos } 15^\circ = \cos (45^\circ - 30^\circ) = \frac{1}{2\sqrt{2}} \cdot (\sqrt{3}+1).$$

35. In the figures attached to the last Article, each of the simple angles A and B was represented as *less* than a right angle. But of whatever magnitude these simple angles are, if the same construction be made, and proper attention be paid to the signs of the sines and cosines of A and B, the same result will invariably be arrived at. For example, let it be required to prove the formula

$$\text{Sin}\,(A - B) = \sin A \,.\, \cos B - \cos A \,.\, \sin B$$

from the annexed figure, where $BAC' = A$, and $C'AD = B$; each angle being greater than a right angle.

From D, any point in AD, draw DC perpendicular to $C'A$ produced; let CEF be perpendicular to AB, DF parallel to AB, DB perpendicular to AB.



Then FB is a parallelogram, and $EF = DB$.

$$\text{Now, Sin}\,(A - B) = \frac{BD}{AD} = \frac{EF}{AD}$$

$$= \frac{CF - CE}{AD}$$

$$= \frac{CF}{CD} \cdot \frac{CD}{AD} - \frac{CE}{AC} \cdot \frac{AC}{AD}$$

$$= \cos FCD \,.\, \sin DAC - \sin CAE \,.\, \cos DAC,$$

$$\text{but}\quad \text{Cos}\, FCD = \cos EAC = \cos C'AB'$$

$$= -\cos\,(180^\circ - C'AB) = -\cos A, \quad \text{Art. 26.}$$

$$\sin DAC = \sin\,(180^\circ - DAC), \text{ Art. 26, } = \sin C'AD = \sin B,$$

$$\sin CAE = \sin C'AB' = \sin A,$$

$$\cos DAC = -\cos\,(180^\circ - DAC) = -\cos B;$$

$$\therefore\ \text{Sin}\,(A - B) = -\cos A \,.\, \sin B + \sin A \,.\, \cos B$$

$$= \sin A \,.\, \cos B - \cos A \,.\, \sin B.$$

36. *If one of the formulæ, as* Sin (A + B) = sin A . cos B + cos A . sin B, *be given, the others may be deduced from it.*

For let B become $(-B)$, then

$$\text{Sin}\,(A - B) = \sin\,\{A + (-B)\} = \sin A \,.\, \cos\,(-B) + \cos A \,.\, \sin\,(-B).$$

$$\text{But}\qquad \text{Cos}\,(-B) = \cos B, \ (23), \qquad \text{and}\quad \text{Sin}\,(-B) = -\sin B, \ (22);$$

$$\therefore\ \text{Sin}\,(A - B) = \sin A \,.\, \cos B - \cos A \,.\, \sin B.$$

Again, $\text{Cos}(A+B) = \sin\{90^\circ - (A+B)\}$, (16) $= \sin\{(90^\circ - A) + (-B)\}$

$$= \sin(90^\circ - A) . \cos(-B) + \cos(90^\circ - A) . \sin(-B)$$

$$= \cos A . \cos B - \sin A . \sin B.$$

Similarly, $\text{Cos}(A-B)$ may be proved $= \cos A . \cos B + \sin A . \sin B$.

37. From the formulæ of (34) the Sine or Cosine of the sum of three or more angles may easily be found in terms of the Sines and Cosines of the simple angles.

Given the Sines and Cosines of the angles A, B, C, *required the Sine of* (A + B + C).

$$\text{Sin}(A+B+C) = \sin\{(A+B)+C\} = \sin(A+B) . \cos C + \cos(A+B) . \sin C$$

$$= (\sin A . \cos B + \cos A . \sin B) . \cos C + (\cos A . \cos B - \sin A . \sin B) . \sin C$$

$$= \sin A . \cos B . \cos C + \sin B . \cos A . \cos C + \sin C . \cos A . \cos B$$

$$- \sin A . \sin B . \sin C \text{..............}(1).$$

In like manner $\text{Sin}(A \pm B \pm C)$, and $\text{Cos}(A \pm B \pm C)$, may be found in terms of the Sines and Cosines of A, B, C; and the same method may be applied to the sum of any number of simple angles.

Cor. If $A + B + C = (2n+1) . 180^\circ$, where n is an integer, since

$$\sin(2n+1)\,180^\circ = 0,$$

the equation (1) becomes

$$\text{Sin} A . \sin B . \sin C = \sin A . \cos B . \cos C$$

$$+ \sin B . \cos A . \cos C + \sin C . \cos A . \cos B \text{............}(2).$$

If $n = 0$, $A + B + C = 180^\circ$, and therefore equation (2) expresses a relation which exists between the sines and cosines of the three angles of any plane triangle.

38. *To shew that* $\text{Sin}\,2A = 2 \sin A . \cos A$.

$$\text{Sin}(A+B) = \sin A . \cos B + \cos A . \sin B,$$

which becomes, by writing A for B,

$$\text{Sin}\,2A = \sin A . \cos A + \cos A . \sin A = 2 \sin A . \cos A.$$

39. *To shew that* $\begin{cases} (1) & \text{Cos}\,2A = \cos^2 A - \sin^2 A. \\ (2) & \text{Cos}\,2A = 2\cos^2 A - 1. \\ (3) & \text{Cos}\,2A = 1 - 2\sin^2 A. \end{cases}$

(1) $$\text{Cos}\,(A+B)=\cos A\,.\,\cos B-\sin A\,.\,\sin B,$$

and writing A for B,

$$\text{Cos}\,2A=\cos A\,.\cos A-\sin A\,.\,\sin A=\cos^2 A-\sin^2 A.$$

Again; $$\text{Cos}\,2A=\cos^2 A-\sin^2 A,$$

$$\text{And}\quad 1=\cos^2 A+\sin^2 A;$$

$$\therefore\ 1+\cos 2A=2\cos^2 A,$$

$$1-\cos 2A=2\sin^2 A.$$

(2) Therefore, $\text{Cos}\,2A=2\cos^2 A-1.$

(3) And, $\text{Cos}\,2A=1-2\sin^2 A.$

40. *To shew that* $\begin{cases}\text{Cos A}+\text{sin A}=\pm\sqrt{(1+\text{sin }2\text{ A})},\\ \text{Cos A}-\text{sin A}=\pm\sqrt{(1-\text{sin }2\text{ A})}.\end{cases}$

Here, $\text{Sin}\,2A=2\sin A\,.\cos A,$ and $1=\cos^2 A+\sin^2 A;$

$\therefore$ by addition and subtraction,

$$1+\sin 2A=\cos^2 A+2\sin A\,.\,\cos A+\sin^2 A,$$

$$1-\sin 2A=\cos^2 A-2\sin A\,.\,\cos A+\sin^2 A;$$

$$\therefore\ \text{Cos}\,A+\sin A=\pm\sqrt{(1+\sin 2A)},$$

$$\text{and}\ \ \text{Cos}\,A-\sin A=\pm\sqrt{(1-\sin 2A)}.$$

41. *To shew that if* A *be less than* 45°,

then $\begin{cases}\text{Cos}\,A=\frac{1}{2}\{\sqrt{(1+\sin 2A)}+\sqrt{(1-\sin 2A)}\},\\ \text{Sin}\,A=\frac{1}{2}\{\sqrt{(1+\sin 2A)}-\sqrt{(1-\sin 2A)}\}.\end{cases}$

By (31) if $A<45^0$, Cos A is $>$ Sin A, and they are both positive; therefore Cos A + sin A and Cos A − sin A are both positive quantities when A is $<45^0$;

$$\therefore\ \text{Cos}\,A+\sin A=+\sqrt{(1+\sin 2A)},$$

$$\text{and}\ \ \text{Cos}\,A-\sin A=+\sqrt{(1-\sin 2A)};$$

whence, by addition and subtraction,

$$2\cos A = \surd(1+\sin 2A) + \surd(1-\sin 2A),$$
$$2\sin A = \surd(1+\sin 2A) - \surd(1-\sin 2A);$$
$$\therefore \begin{cases} \text{Cos}\, A = \frac{1}{2}\{\surd(1+\sin 2A) + \surd(1-\sin 2A)\}, \\ \text{Sin}\, A = \frac{1}{2}\{\surd(1+\sin 2A) - \surd(1-\sin 2A)\}. \end{cases}$$

42. *The equations deduced in* Art. 40 *can be applied in any case to determine the Sine and Cosine of* A *from the Sine of* 2 A.

For example, (1) If A be an angle $> 3 \times 45^0$ but $< 4 \times 45^0$, (*i.e.* if it be any angle comprehended under the form $180^0 - B$, where B is $< 45^0$,) Cos A is negative, and greater in magnitude than Sin A, which is a positive quantity.

In this case therefore,

$$\text{Cos}\, A + \sin A = -\surd(1+\sin 2A),$$
$$\text{Cos}\, A - \sin A = -\surd(1-\sin 2A);$$
$$\therefore\ \text{Cos}\, A = -\tfrac{1}{2}\{\surd(1+\sin 2A) + \surd(1-\sin 2A)\},$$
$$\text{Sin}\, A = \ \tfrac{1}{2}\{\surd(1-\sin 2A) - \surd(1+\sin 2A)\}.$$

If at first sight this value of Sin A appear to be negative, it is to be remembered that Sin $2A$ is a negative quantity, ($2A$ being between 270^0 and 360^0) and therefore $1-\sin 2A$ is greater than $1+\sin 2A$; wherefore the value of sin A is here a positive quantity, as it ought to be.

So (2), If A be a negative angle which is between -45^0 and -90^0, Cos A is a positive quantity, and is less in magnitude than Sin A, which is a negative quantity; wherefore the equations of Art. 40 to be taken, are,

$$\text{Cos}\, A + \sin A = -\surd(1+\sin 2A),$$
$$\text{Cos}\, A - \sin A = +\surd(1-\sin 2A).^*$$

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* *To determine the limits between which the values of* A *must lie, which satisfy the equations,*

$$\text{Sin}\, A + \cos A = -\surd(1+\cos 2A),$$
$$\text{Cos}\, A - \sin A = -\surd(1-\cos 2A).$$

For positive values of A,

The former equation is fulfilled if A be between $90^0 + 45^0$ and 180^0: for the value of Cos A, which is negative, is greater for such angles than the value of Sin A, which is positive. So A may lie between 180^0 and 270^0; and between 270^0 and $270^0 + 45^0$.

Wherefore if A be between 135^0 and 315^0 the former equation is fulfilled. And it may be shewn, in like manner, that the latter equation is fulfilled if A lie between 45^0 and 315^0.

So that both equations will be fulfilled if A be between 135^0 and 315^0. And as the Sine and Cosine of any angle remain the same if the angle itself be increased by 360^0, it follows that all the positive values of A which satisfy both the equations, lie between $n \,.\, 360^0 + 135^0$ and $n \,.\, 360^0 + 315^0$, (i. e. between $(8n+3) \,.\, 45^0$ and $(8n+7) \,.\, 45^0$), where n is 0 or any positive integer.

And in the same manner

43. *Given the Tangents of two angles, to find the Tangents of their Sum and their Difference.*

$$\operatorname{Tan}(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\,.\cos B+\cos A\,.\sin B}{\cos A\,.\cos B-\sin A\,.\sin B},$$

and dividing the numerator and the denominator by $\cos A\,.\cos B$,

$$\operatorname{Tan}(A+B)=\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin A}{\cos A}\,.\,\dfrac{\sin B}{\cos B}}=\frac{\tan A+\tan B}{1-\tan A\,.\tan B}.$$

$$\text{Similarly,}\quad \operatorname{Tan}(A-B)=\frac{\tan A-\tan B}{1+\tan A\,.\tan B}.$$

COR. 1. If $B=A$, $\quad \operatorname{Tan}2A=\dfrac{2\tan A}{1-\tan^2 A}.$

COR. 2. If $B=45^\circ$, since $\operatorname{Tan}45^\circ=1\ldots\ldots(32)$

$$\therefore\ \operatorname{Tan}(A+45^\circ)=\frac{\tan A+1}{1-\tan A}\ldots\ldots\ldots\ldots(1).$$

$$=\frac{\dfrac{\sin A}{\cos A}+1}{1-\dfrac{\sin A}{\cos A}}=\frac{\sin A+\cos A}{\cos A-\sin A}\ldots\ldots\ldots\ldots(2).$$

COR. 3. Similarly, $\quad \operatorname{Tan}(A-45^\circ)=\dfrac{\tan A-1}{\tan A+1}\ldots\ldots\ldots\ldots(3);$

$$\text{or}\quad =\frac{\sin A-\cos A}{\sin A+\cos A}\ldots\ldots\ldots\ldots(4);$$

COR. 4. $\operatorname{Tan}(A+45^\circ)+\tan(A-45^\circ)=\dfrac{\tan A+1}{1-\tan A}+\dfrac{\tan A-1}{1+\tan A}$

$$=\frac{4\tan A}{1-\tan^2 A},\ =2\tan 2A,\ \text{by Cor. 1}\ldots\ldots\ldots\ldots(5).$$

If A be $<45^\circ$;

Since $\operatorname{Tan}(A-45^\circ)=\tan-(45^\circ-A)=-\tan(45^\circ-A)$, (23),

the last expression becomes,

$$\operatorname{Tan}(45^\circ+A)-\tan(45^\circ-A)=2\tan 2A\ldots\ldots\ldots\ldots(6).$$

And in the same manner the negative values of A which satisfy both the equations may be shewn to lie between $m\,.\,360^\circ-45^\circ$ and $m\,.\,360^\circ-225^\circ$, m being 0 or any negative integer;

i. e. between $(8m-1)\,.\,45^\circ$ and $(8m-5)\,.\,45^\circ$,

$\ldots\ldots\ldots\ldots\{8(m-1)+7\}\,.\,45^\circ$ and $\{8(m-1)+3\}\,.\,45^\circ$,

(and if n be written for $m-1$),

between $(8n+7)\,.\,45^\circ$, and $(8n+3)\,.\,45^\circ$;

which are of the same form as the limits obtained for the positive values of A.

Wherefore the required limits of the values of A are $(8n+7)\,.\,45^\circ$ and $(8n+3)\,.\,45^\circ$; where n may be 0, or any integer, positive or negative.

44. *To shew that* $\dfrac{\text{Tan}\,A+\tan B}{\text{Tan}\,A-\tan B}=\dfrac{\sin(A+B)}{\sin(A-B)}$.

$$\frac{\text{Tan}\,A+\tan B}{\text{Tan}\,A-\tan B}=\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{\dfrac{\sin A}{\cos A}-\dfrac{\sin B}{\cos B}}$$

$$=\frac{\sin A.\cos B+\cos A.\sin B}{\sin A.\cos B-\cos A.\sin B}=\frac{\sin(A+B)}{\sin(A-B)}.$$

45. *Given* Tan A, Tan B, Tan C, *to find* Tan (A + B + C).

$$\text{Tan}(A+B+C)=\tan\{(A+B)+C\}=\frac{\tan(A+B)+\tan C}{1-\tan(A+B).\tan C},\quad(43),$$

$$=\frac{\dfrac{\tan A+\tan B}{1-\tan A.\tan B}+\tan C}{1-\dfrac{\tan A+\tan B}{1-\tan A.\tan B}.\tan C}$$

$$=\frac{\tan A+\tan B+\tan C-\tan A.\tan B.\tan C}{1-\tan A.\tan B-\tan A.\tan C-\tan B.\tan C}.$$

In the same manner the tangent of the sum of four or more angles might be found in terms of the tangents of the simple angles.

Cor. If $A+B+C=(2n+1).180^0$, n being an integer, $\text{Tan}(A+B+C)=0$;

$$\therefore\ \tan A+\tan B+\tan C-\tan A.\tan B.\tan C=0,$$

$$\text{or,}\quad \text{Tan}\,A+\tan B+\tan C=\tan A.\tan B.\tan C.$$

And since if $n=0$, $A+B+C=180^0$, this relation between the tangents of A, B, C, is one which exists between the tangents of the angles of any plane triangle.

46. *To find the values of* Sin 2 A, *and* Cos 2 A, *in terms of* Tan A.

$$\text{Sin}\,2A=2\sin A.\cos A,\quad(38),\qquad=\frac{2\sin A}{\cos A}.\cos^2 A$$

$$=\frac{2\tan A}{\sec^2 A},\quad(27.\ 2),\qquad=\frac{2\tan A}{1+\tan^2 A},\quad(27.\ 7).$$

Again, $\quad\text{Cos}\,2A=2\cos^2 A-1,\quad(39.\ 2),\qquad=\dfrac{2}{\sec^2 A}-1$

$$=\frac{2}{1+\tan^2 A}-1=\frac{1-\tan^2 A}{1+\tan^2 A}.$$

47. The following values of $\text{Sin}\, 2A$, $\text{Cos}\, 2A$, $\text{Tan}\, 2A$, are of frequent occurrence, and necessary to be remembered; those which have not been proved already are easily found after the method of the last Article.

1. $\text{Sin}\, 2A = 2 \sin A \,.\, \cos A.$

2. $\ldots\ldots = \dfrac{2 \tan A}{1 + \tan^2 A}.$

3. $\ldots\ldots = \dfrac{2 \,.\, \sqrt{(\sec^2 A - 1)}}{\sec^2 A}.$

4. $\text{Tan}\, 2A = \dfrac{2 \tan A}{1 - \tan^2 A}.$

1. $\text{Cos}\, 2A = \cos^2 A - \sin^2 A.$

2. $\ldots\ldots = 2 \cos^2 A - 1.$

3. $\ldots\ldots = 1 - 2 \sin^2 A.$

4. $\ldots\ldots = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}.$

5. $\ldots\ldots = \dfrac{2 - \sec^2 A}{\sec^2 A}.$

48. In the same way, the following values of $\text{Sin}\, 2A$ and $\text{Cos}\, 2A$ can be found in terms of $\text{Cot}\, A$, $\text{Cosec}\, A$, and $\text{Versin}\, A$.

$$\text{Sin}\, 2A = \frac{2 \cot A}{1 + \cot^2 A}, \quad \text{or} = \frac{2\sqrt{(\text{cosec}^2 A - 1)}}{\text{cosec}^2 A},$$

$$\text{or} = 2 \,.\, (1 - \text{versin}\, A) \,.\, \sqrt{(2 \,\text{vers}\, A - \text{vers}^2 A)}.$$

$$\text{Cos}\, 2A = \frac{\cot^2 A - 1}{\cot^2 A + 1}, \quad \text{or} = \frac{\text{cosec}^2 A - 2}{\text{cosec}^2 A}, \quad \text{or} = 1 - 2 \,.\, (2 \,\text{vers}\, A - \text{vers}^2 A).$$

49. The easiest method of deducing formulæ such as these, is first to express $\text{Sin}\, 2A$ and $\text{Cos}\, 2A$ in terms of $\text{Sin}\, A$ and $\text{Cos}\, A$.

Thus let it be required to prove that $\text{Cos}\, 2A = \dfrac{\text{cosec}^2 A - 2}{\text{cosec}^2 A}$.

Since $\text{Cos}\, 2A = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A$; and $\text{Sin}\, A = \dfrac{1}{\text{cosec}\, A}$, (27);

$$\therefore\ \text{Cos}\, 2A = 1 - \frac{2}{\text{cosec}^2 A} = \frac{\text{cosec}^2 A - 2}{\text{cosec}^2 A}.$$

50. Since $\text{Sin}\, (A + B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B,$

and $\text{Sin}\, (A - B) = \sin A \,.\, \cos B - \cos A \,.\, \sin B;$

∴ by adding and subtracting,

$$\text{Sin}\,(A+B)+\sin\,(A-B)=2\sin A\,.\cos B\,\ldots\ldots\ldots(1).$$

$$\text{Sin}\,(A+B)-\sin\,(A-B)=2\cos A\,.\sin B\,\ldots\ldots\ldots(2).$$

Similarly,

$$\text{Cos}\,(A+B)+\cos\,(A-B)=2\cos A\,.\cos B\,\ldots\ldots\ldots(3).$$

$$\text{Cos}\,(A-B)-\cos\,(A+B)=2\sin A\,.\sin B\,\ldots\ldots\ldots(4).$$

51. *To find the values of* Sin A ± sin B, *and* Cos A ± cos B, *in terms of the Sines and Cosines of* $\frac{1}{2}$ (A + B) *and* $\frac{1}{2}$ (A − B).

Since

$$A=\tfrac{1}{2}(A+B)+\tfrac{1}{2}(A-B),\quad \text{and}\quad B=\tfrac{1}{2}(A+B)-\tfrac{1}{2}(A-B);$$

$$\therefore\ \text{Sin}\,A=\sin\tfrac{1}{2}(A+B)\,.\cos\tfrac{1}{2}(A-B)+\cos\tfrac{1}{2}(A+B)\,.\sin\tfrac{1}{2}(A-B),$$

$$\text{Sin}\,B=\sin\tfrac{1}{2}(A+B)\,.\cos\tfrac{1}{2}(A-B)-\cos\tfrac{1}{2}(A+B)\,.\sin\tfrac{1}{2}(A-B);$$

$$\therefore\ \text{Sin}\,A+\sin B=2\sin\tfrac{1}{2}(A+B)\,.\cos\tfrac{1}{2}(A-B)\,\ldots\ldots\ldots(1).$$

$$\text{Sin}\,A-\sin B=2\cos\tfrac{1}{2}(A+B)\,.\sin\tfrac{1}{2}(A-B)\,\ldots\ldots\ldots(2).$$

Similarly

$$\text{Cos}\,A+\cos B=2\cos\tfrac{1}{2}(A+B)\,.\cos\tfrac{1}{2}(A-B)\,\ldots\ldots\ldots(3).$$

$$\text{Cos}\,B-\cos A=2\sin\tfrac{1}{2}(A+B)\,.\sin\tfrac{1}{2}(A-B)\,\ldots\ldots\ldots(4).$$

These four formulæ (*which are of the very greatest utility*) might have been deduced from the formulæ of the last Article, by making $A+B=S$ and $A-B=D$; in which case $A=\frac{1}{2}(S+D)$ and $B=\frac{1}{2}(S-D)$.

52. Dividing (2) of Art. 51 by (1),

$$\frac{\text{Sin}\,A-\sin B}{\text{Sin}\,A+\sin B}=\frac{2\cos\frac{1}{2}(A+B)\,.\sin\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A+B)\,.\cos\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}.$$

Similarly, dividing (4) by (3),

$$\frac{\text{Cos}\,B-\cos A}{\text{Cos}\,A+\cos B}=\tan\tfrac{1}{2}(A+B)\,.\tan\tfrac{1}{2}(A-B).$$

So also, $\dfrac{\text{Sin}\,A\pm\sin B}{\text{Cos}\,A+\cos B}=\tan\frac{1}{2}(A\pm B)$; and $\dfrac{\text{Sin}\,A\pm\sin B}{\text{Cos}\,B-\cos A}=\text{cotan}\,\frac{1}{2}(A\mp B)$.

(The upper of the double signs going together, and the lower together).

53. $\text{Tan}\, A \pm \tan B = \dfrac{\sin A}{\cos A} \pm \dfrac{\sin B}{\cos B}$

$$= \frac{\sin A \, . \cos B \pm \cos A \, . \sin B}{\cos A \, . \cos B} = \frac{\sin(A \pm B)}{\cos A \, . \cos B}.$$

Similarly, $\text{Cot}\, B \pm \cot A = \dfrac{\sin(A \pm B)}{\sin A \, . \sin B}$.

(The upper signs going together, and the lower together).

54. $\text{Sin}\,(A+B) \, . \sin(A-B) = \sin^2 A \, . \cos^2 B - \cos^2 A \, . \sin^2 B$

$$= \sin^2 A \, . (1 - \sin^2 B) - (1 - \sin^2 A) \, . \sin^2 B$$

$$= \sin^2 A - \sin^2 B.$$

Similarly, $\text{Sin}\,(A+B) \, . \sin(A-B) = \cos^2 B - \cos^2 A.$

And in like manner it may be shewn that

$\text{Cos}\,(A+B) \, . \cos(A-B) = \cos^2 A - \sin^2 B$; or $= \cos^2 B - \sin^2 A.$

55. *To prove that*

$$\begin{cases} \text{Sin}\, nA + \sin(n-2)A = 2 \sin(n-1)A \, . \cos A, \\ \text{Cos}\, nA + \cos(n-2)A = 2 \cos(n-1)A \, . \cos A. \end{cases}$$

$\text{Sin}\, nA = \sin\{(n-1)A + A\} = \sin(n-1)A \, . \cos A + \cos(n-1)A \, . \sin A.$

So, $\text{Sin}\,(n-2)A = \sin(n-1)A \, . \cos A - \cos(n-1)A \, . \sin A$;

$\therefore \text{Sin}\, nA + \sin(n-2)A = 2 \sin(n-1)A \, . \cos A$..................(1).

Again; in the same manner,

$\text{Cos}\, nA = \cos(n-1)A \, . \cos A - \sin(n-1)A \, . \sin A,$

and $\text{Cos}\,(n-2)A = \cos(n-1)A \, . \cos A + \sin(n-1)A \, . \sin A$;

$\therefore \text{Cos}\, nA + \cos(n-2)A = 2 \cos(n-1)A \, . \cos A$......................(2).

Cor. If $n=2$, then, from (1), $\text{Sin}\, 2A = 2 \sin A \, . \cos A,$

.... from (2), $\text{Cos}\, 2A + 1 = 2\cos^2 A$, or, $\text{Cos}\, 2A = 2\cos^2 A - 1.$

If $n=3$; From (1), $\text{Sin}\, 3A = 2 \sin 2A \, . \cos A - \sin A = 4 \sin A \, . \cos^2 A - \sin A$

$$= 4 \sin A \, . (1 - \sin^2 A) - \sin A = 3 \sin A - 4 \sin^3 A,$$

From (2), $\text{Cos}\, 3A = 2 \cos 2A \, . \cos A - \cos A$

$$= 2 \cos A \, . (2\cos^2 A - 1) - \cos A = 4\cos^3 A - 3 \cos A;$$

and by successive substitutions, $\text{Sin}\, 4A$, $\text{Sin}\, 5A$... $\text{Cos}\, 4A$, $\text{Cos}\, 5A$... might be found.

56. In like manner,

$\text{Sin}\, nA - \sin(n-2)A = 2 \cos(n-1)A \, . \sin A$.........(1),

$\text{Cos}\,(n-2)A - \cos nA = 2 \sin(n-1)A \, . \sin A$.........(2).

57. From the latter of these formulæ, by means of successive substitutions, the Cosine of nA may be found in terms of the Sines of A and its multiples.

Suppose n to be even, and $= 2m$.

$$\therefore \text{Cos}\, 2(m-1)A - \cos 2mA = 2\sin(2m-1)A \,.\, \sin A,$$

$$\text{so} \quad \cos 2(m-2)A - \cos 2(m-1)A = 2\sin(2m-3)A \,.\, \sin A,$$

$$\cos 2(m-3)A - \cos 2(m-2)A = 2\sin(2m-5)A \,.\, \sin A,$$

$$\&\text{c.} \qquad = \qquad \&\text{c.}$$

$$\cos 2(m-m)A - \cos 2\{m-(m-1)\}A = 2\sin A \,.\, \sin A,$$

Whence, by addition,

(since $\cos 2(m-m)A$, or $\cos 0$, $= 1$,)

$$1 - \cos 2mA = 2\sin A \,.\, \{\sin(2m-1)A + \sin(2m-3)A + \ldots + \sin 3A + \sin A\}.$$

$$\therefore \text{Cos}\, 2mA = 1 - 2\sin A \,.\, \{\sin(2m-1)A + \sin(2m-3)A + \ldots + \sin 3A + \sin A\} \ldots (1).$$

So if n were odd, and $= 2m+1$, it would appear that

$$\text{Cos}\,(2m+1)A = \cos A - 2\sin A \,.\, \{\sin 2mA + \sin 2(m-1)A + \ldots + \sin 4A + \sin 2A\} \ldots (2).$$

Cor. If $m = 1$, these formulæ give,

$$\text{Cos}\, 2A = 1 - 2\sin A \,.\, \sin A = 1 - 2\sin^2 A.$$

$$\text{Cos}\, 3A = \cos A - 2\sin A \,.\, \sin 2A = \cos A - 2\sin A \,.\, 2\sin A \,.\, \cos A$$

$$= \cos A - 4\cos A \,.\, (1 - \cos^2 A) = 4\cos^3 A - 3\cos A.$$

58. *To find the Sines and Cosines of* 18°, 72°, 36°, *and* 54°.

$$\text{Sin}\, 36^\circ = \cos(90^\circ - 36^\circ) = \cos 54^\circ,$$

$$\text{or, if } 18^\circ = A, \qquad \sin 2A = \cos 3A;$$

$$\therefore 2\sin A \,.\, \cos A = 2\cos 2A \,.\, \cos A - \cos A, \quad (55);$$

$$\therefore 2\sin A = 2\cos 2A - 1 = 2\,.\,(1 - 2\sin^2 A) - 1;$$

$$\therefore 4\sin^2 A + 2\sin A = 1.$$

And, solving this equation, $\text{Sin}\, A = \dfrac{\pm\sqrt{5} - 1}{4}$; of which the positive sign is to be taken, because $\sin 18^\circ$ is a positive quantity.

$$\therefore \tfrac{1}{4}(\sqrt{5} - 1) = \sin 18^\circ = \cos(90^\circ - 18^\circ) = \cos 72^\circ \ldots\ldots\ldots\ldots (1).$$

And $$\text{Cos}^2 18^\circ = 1 - \sin^2 18^\circ = 1 - \frac{6-2\sqrt{5}}{16} = \frac{10+2\sqrt{5}}{16},$$

$$\therefore\ \text{Cos}\, 18^\circ = \tfrac{1}{4}\sqrt{(10+2\sqrt{5})} = \sin 72^\circ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Again; $$\text{Sin}\, 54^\circ = \cos 36^\circ = \cos 2 \times 18^\circ = \cos^2 18^\circ - \sin^2 18^\circ$$

$$= \frac{10+2\sqrt{5}}{16} - \frac{6-2\sqrt{5}}{16}$$

$$= \tfrac{1}{4}(1+\sqrt{5}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

$$\text{Cos}^2 54^\circ = 1 - \sin^2 54^\circ = 1 - \frac{6+2\sqrt{5}}{16} = \frac{1}{16}(10-2\sqrt{5}),$$

$$\therefore\ \text{Cos}\, 54^\circ = \tfrac{1}{4}\sqrt{(10-2\sqrt{5})} = \sin 36^\circ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

59. *If an angle receive any increment, to find the corresponding increment of the Sine of the angle.*

Let the angle A receive an increment a, and let the corresponding increment of the Sine of A be represented by $\Delta \sin A$.

Then, $$\Delta \sin A = \sin(A + a) - \sin A$$

$$= \sin A \,.\, \cos a + \cos A \,.\, \sin a - \sin A$$

$$= \cos A \,.\, \sin a - \sin A\,(1 - \cos a)$$

$$= \cos A \,.\, \sin a \,.\, \left(1 - \tan A \,.\, \frac{2\sin^2 \frac{1}{2} a}{\sin a}\right) \ldots\ldots (39).$$

$$= \cos A \,.\, \sin a \,.\, \left(1 - \tan A \,.\, \frac{2\sin^2 \frac{1}{2} a}{2\sin \frac{1}{2} a \,.\, \cos \frac{1}{2} a}\right)$$

$$= \cos A \,.\, \sin a \,.\, (1 - \tan A \,.\, \tan \tfrac{1}{2} a).$$

Cor. If a be very small, Tan $\frac{1}{2}a$ is very small. In this case, if Tan A be not exceedingly large, (that is, *if A be not nearly equal to* $\overline{2n+1} \,.\, 90^\circ$, n being an integer,) $\tan A \,.\, \tan \frac{1}{2} a$ is a very small quantity, and may be neglected in comparison with unity. *When, therefore, a is very small, and A is also not nearly equal to* $(2n+1)\, 90^\circ$,

$$\Delta \sin A = \cos A \,.\, \sin a, \text{ very nearly.}$$

Hence it appears that when A is an angle of a *triangle,* this result cannot be applied to determine the corresponding increment of Sin A which results from A receiving a *small* given increment, if A be nearly equal to a right angle.

60. *If an angle receive any increment, to find the corresponding decrement of the Cosine of the angle.*

$$\Delta \cos A = \cos (A + \alpha) - \cos A = \cos A \,.\, \cos \alpha - \sin A \,.\, \sin \alpha - \cos A$$

$$= - \sin A \,.\, \sin \alpha - \cos A \,.\, (1 - \cos \alpha) = - \sin A \,.\, \sin \,.\, \left(1 + \cot A \,.\, \frac{2 \sin^2 \frac{1}{2} \alpha}{\sin \alpha}\right)$$

$$= - \sin A \,.\, \sin \alpha \,.\, (1 + \cot A \,.\, \tan \tfrac{1}{2} \alpha).$$

Cor. 1. And as before, if α be very small, and also Cot A be not exceedingly large, (i. e. *if A be not $= 2n \,.\, 90^0$ nearly*), $\cot A \,.\, \tan \frac{1}{2} \alpha$ may be neglected with respect to unity, and

$$\Delta \cos A = - \sin A \,.\, \sin \alpha, \text{ very nearly.}$$

Note. It must be carefully borne in mind, that unless (1), α be a very small angle, and, (2), A be, besides, an angle which is not nearly equal to 0° or 180°, this result cannot be applied to any particular case where A is an angle of a triangle.

Cor. 2. If A be less than 90°, Cos A is positive, and Sin A being also positive, in this case $\Delta \cos A$ is necessarily negative. That is, in angles less than a right angle, as the angle increases its cosine decreases.

If A be greater than one right angle but less than two, Cos A is negative, and, Sin A being positive, $\Delta \cos A$ is negative. That is, when the angle is greater than one right angle, but less than two, as the angle increases, the cosine also increases in magnitude, but is negative.

61. *If an angle receive any increment, to find the corresponding increment of the Secant of the angle.*

$$\Delta \sec A = \sec (A + \alpha) - \sec A = \frac{1}{\cos (A + \alpha)} - \frac{1}{\cos A} = \frac{\cos A - \cos (A + \alpha)}{\cos A \,.\, \cos (A + \alpha)} .$$

$$= \frac{\sin A \,.\, \sin \alpha \,.\, \{1 + \cot A \,.\, \tan \frac{1}{2} \alpha\} \text{......(60)}}{\cos A \,.\, (\cos A \,.\, \cos \alpha - \sin A \,.\, \sin \alpha) \text{......(34)}}$$

$$= \frac{\sin A}{\cos A} \cdot \frac{1}{\cos A} \cdot \frac{\sin \alpha}{\cos \alpha} \cdot \frac{1 + \cot A \,.\, \tan \frac{1}{2} \alpha}{1 - \tan A \,.\, \tan \alpha} = \tan A \,.\, \sec A \,.\, \tan \alpha \,.\, \frac{1 + \cot A \,.\, \tan \frac{1}{2} \alpha}{1 - \tan A \,.\, \tan \alpha} .$$

Cor. If α be very small, and neither Tan A nor Cot A be very large (that is, *if A be not nearly equal to $n \,.\, 90^0$*), $\cot A \,.\, \tan \frac{1}{2} \alpha$ and $\tan A \,.\, \tan \alpha$ will both be so small that they may be neglected when compared with unity. In this case therefore

$$\Delta \sec A = \tan A \,.\, \sec A \,.\, \tan \alpha, \text{ very nearly.}$$

Note. It is to be remarked that before this result can be applied in any case where A is an angle of a triangle, α must be a very small angle, and A must moreover not be nearly equal to 0, 90°, or 180°.

62. *If an angle receive any increment, to find the corresponding increment of the Tangent of the angle.*

$$\Delta \tan A = \tan(A+\alpha) - \tan A = \frac{\sin(A+\alpha)}{\cos(A+\alpha)} - \frac{\sin A}{\cos A}$$

$$= \frac{\sin(A+\alpha).\cos A - \cos(A+\alpha).\sin A}{\cos A.(\cos A.\cos\alpha - \sin A.\sin\alpha)}.$$

But $\sin(A+\alpha).\cos A - \cos(A+\alpha).\sin A = \sin\{(A+\alpha)-A\} = \sin\alpha$;

$$\therefore\ \Delta \tan A = \frac{\sin\alpha}{\cos^2 A.\cos\alpha.(1-\tan A.\tan\alpha)} = \sec^2 A.\tan\alpha.\frac{1}{1-\tan A.\tan\alpha}.$$

Cor. If α be very small, and also $\operatorname{Tan} A$ be not very large, (that is, *if* A *be not* $(2n+1)\,90°$ *nearly*), then

$$\Delta \tan A = \sec^2 A.\tan\alpha, \text{ very nearly.}$$

Note. If A be an angle of a triangle, this result is not true when A is nearly equal to a right angle.

63. *For a given small increment of* A, *the increment of the Sine of the angle is* >, =, < *the decrement of the Cosine, according as* Cos A *is* >, =, < Sin A; A *not being very small, or nearly a multiple of* 90°.

For $\Delta \sin A = \cos A.\sin\alpha$, if A be not nearly $(2n+1)\,90°$; Art. 59. Cor.

$\Delta \cos A = -\sin A.\sin\alpha$, if A be not nearly $2n.90°$; Art. 60. Cor. 1.

Wherefore (n being 0, or any integer), if A be not very small, or nearly a multiple of 90°,

$$\Delta \sin A \text{ is } >, =, < (-\Delta\cos A), \text{ as } \cos A \text{ is } >, =, < \sin A.$$

Cor. In angles less than 90°, $\Delta \sin A$ is $>$ or $<$ $(-\Delta\cos A)$

as A is $<$ or $>$ 45°...(31).

64. **Def.** By $\text{Tan}^{-1}t$ the angle is indicated of which the tangent is t; *i. e.* if $t = \tan A$, then $A = \tan^{-1}t$.

So $\text{Sin}^{-1}s$, and $\text{Cos}^{-1}c$, &c. respectively indicate the angle of which the sine is s, and that of which the cosine is c, &c.

65. *To shew that* $\text{Tan}^{-1}t_1 + \tan^{-1}t_2 = \tan^{-1}\dfrac{t_1 + t_2}{1 - t_1 t_2}$, *and*

$$\text{Tan}^{-1}t_1 - \tan^{-1}t_2 = \tan^{-1}\frac{t_1 - t_2}{1 + t_1 t_2}.$$

Let $\text{Tan}\,A = t_1$, and $\text{Tan}\,B = t_2$.

Then, by definition, $A = \tan^{-1}t_1$, and $B = \tan^{-1}t_2$.

$$\text{Now}\quad \text{Tan}\,(A + B) = \frac{\tan A + \tan B}{1 - \tan A \,.\, \tan B};$$

$$\therefore \text{ by def. } A + B = \tan^{-1}\frac{\tan A + \tan B}{1 - \tan A \,.\, \tan B};$$

$$\text{Or,}\quad \text{Tan}^{-1}t_1 + \tan^{-1}t_2 = \tan^{-1}\frac{t_1 + t_2}{1 - t_1 t_2} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\text{Similarly,}\quad \text{Tan}^{-1}t_1 - \tan^{-1}t_2 = \tan^{-1}\frac{t_1 - t_2}{1 + t_1 t_2} \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

66. *If* t_1, t_2, ... t_n *be the tangents of any angles, then*

$$\text{Tan}^{-1}t_1 - \tan^{-1}t_n = \tan^{-1}\frac{t_1 - t_2}{1 + t_1 t_2} + \tan^{-1}\frac{t_2 - t_3}{1 + t_2 t_3} + \ldots\ldots + \tan^{-1}\frac{t_{n-1} - t_n}{1 + t_{n-1} t_n}.$$

$$\text{For,}\quad \text{Tan}^{-1}t_1 - \tan^{-1}t_2 = \tan^{-1}\frac{t_1 - t_2}{1 + t_1 t_2},$$

$$\tan^{-1}t_2 - \tan^{-1}t_3 = \tan^{-1}\frac{t_2 - t_3}{1 + t_2 t_3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\tan^{-1}t_{n-1} - \tan^{-1}t_n = \tan^{-1}\frac{t_{n-1} - t_n}{1 + t_{n-1} t_n};$$

$\therefore$ by addition, $\text{Tan}^{-1}t_1 - \tan^{-1}t_n$

$$= \tan^{-1}\frac{t_1 - t_2}{1 + t_1 t_2} + \tan^{-1}\frac{t_2 - t_3}{1 + t_2 t_3} + \ldots + \tan^{-1}\frac{t_{n-1} - t_n}{1 + t_{n-1} t_n}.$$

67. Examples of questions solved by the application of formulæ proved in this Chapter and the preceding.

(1) *To prove that* $\frac{\text{Cos A} + \sin \text{A}}{\text{Cos A} - \cos \text{A}} = \sec 2\text{A} + \tan 2\text{A}$.

[It is here required to bring the proposed fraction into one of which denominator shall be $\cos 2A$, or $\cos^2 A - \sin^2 A$. Multiply, therefore, the numerator and denominator by the numerator, and]

$$\frac{\text{Cos } A + \sin A}{\text{Cos } A - \sin A} = \frac{(\cos A + \sin A)^2}{\cos^2 A - \sin^2 A} = \frac{\cos^2 A + \sin^2 A + 2\cos A . \sin A}{\cos^2 A - \sin^2 A}$$

$$= \frac{1 + \sin 2A}{\cos 2A}; \text{ by (27. 6), and (38), (39).}$$

$$= \frac{1}{\cos 2A} + \frac{\sin 2A}{\cos 2A} = \sec 2A + \tan 2A.$$

(2) *To prove that* $\text{Cos } 2\text{A} = \frac{1}{1 + \tan 2\text{A} . \tan \text{A}}$.

[The equation becomes, when inverted, $\frac{1}{\cos 2A} = 1 + \tan 2A . \tan A$; where if the right-hand side were expressed in terms of the sines and cosines of A and $2A$, and then put into a fractional form, the denominator would be $\cos 2A . \cos A$, and the numerator would involve the sines and cosines of the same angles. The first object therefore is to express $\frac{1}{\cos 2A}$ in a fraction of such a form.]

$$\text{Now, } \frac{1}{\text{Cos } 2A} = \frac{\cos A}{\cos 2A . \cos A} = \frac{\cos(2A - A)}{\cos 2A . \cos A}$$

$$= \frac{\cos 2A . \cos A + \sin 2A . \sin A}{\cos 2A . \cos A} = 1 + \frac{\sin 2A}{\cos 2A} . \frac{\sin A}{\cos A} = 1 + \tan 2A . \tan A;$$

$$\therefore \text{ Cos } 2A = \frac{1}{1 + \tan 2A . \tan A}.$$

Note. In the following Examples it is required to determine an angle from some given relation between its Goniometrical Ratios and those of either a multiple of the angle sought, or of some given angle; and conversely.

(3) *Determine* A *from the equation,* $\text{Sin } 2\text{A} = \sin \text{A}$.

$$\text{Sin } A = \sin 2A = 2\sin A . \cos A \text{............(38);}$$

$$\therefore\ 2\cos A = 1, \text{ and } \cos A = \tfrac{1}{2}. \text{ Wherefore } A = 60^\circ \text{............(32).}$$

(4) *Determine* B *from the equation,*

$$\text{Sin A} + \sin(2\,\text{B} + \text{A}) - \sin(2\,\text{B} - \text{A}) = \sin(\text{B} + \text{A}) - \sin(\text{B} - \text{A});$$

$$\therefore\ \sin A + 2\cos 2B \,.\, \sin A = 2\cos B \,.\, \sin A; \quad \text{Art. 50, (2).}$$

$$\therefore\ 1 + 2\cos 2B = 2\cos B;$$

$$\therefore\ 1 + 2\,.\,(2\cos^2 B - 1) = 2\cos B \ldots\ldots\ldots\ldots (39).$$

$$\text{Whence}\quad \text{Cos}\, B = \tfrac{1}{4}(1 \pm \surd 5);$$

$$\text{and, Art. 58, (3),}\quad \tfrac{1}{4}(1 + \surd 5) = \cos 36^\circ.$$

$$\text{Art. 58, (1),}\quad \tfrac{1}{4}(1 - \surd 5) = -\tfrac{1}{4}(\surd 5 - 1) = -\cos 72^\circ = \cos(180^\circ - 72^\circ) = \cos 108^\circ.$$

$$\therefore\ B \text{ is } 36^\circ, \text{ or } 108^\circ.$$

(5) *To prove that* $2\cos 11^\circ, 15' = \sqrt{2 + \sqrt{\{2 + \sqrt{2}\}}}$.

$$\text{Cos } 45^\circ = \frac{1}{\sqrt{2}}; \quad \therefore\ 2\cos 45^\circ = \sqrt{2},$$

$$\therefore\ 2\,.\,\{2\cos^2 \tfrac{1}{2}\,45^\circ - 1\} = \sqrt{2},$$

$$\therefore\ 2\cos \tfrac{1}{2}\,45^\circ = \sqrt{2 + \sqrt{2}};$$

$$\therefore\ 2\,.\,\{2\cos^2 \frac{45^\circ}{2^2} - 1\} = \sqrt{2 + \sqrt{2}};$$

$$\therefore\ 2\cos \frac{45^\circ}{2^2}, \quad \text{or}\quad 2\cos 11^\circ, 15' = \sqrt{2 + \sqrt{2 + \sqrt{2}}}.$$

Cor. By repeating the same process n times, it would appear in like manner, that

$$2\cos \frac{45^\circ}{2^n} = \sqrt{2 + \sqrt{(2 + \&c.)}}$$

where 2, with the sign of a square root over it, appears $n + 1$ time in the second member of the equation, the square root every time reaching to the end of the expression.

(6) *If* $x \,.\, \tan A = (\sqrt{1+x} - 1) \,.\, (\sqrt{1-x} + 1)$, *required to prove that* $x = \sin 4A$.

$$x \,.\, \tan A = (\sqrt{1+x} - 1) \,.\, (\sqrt{1-x} + 1) \,.\, \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} = x \,.\, \frac{\sqrt{1-x}+1}{\sqrt{1+x}+1};$$

$$\therefore \;\; \sqrt{1-x} + 1 = (\sqrt{1+x} + 1) \,.\, \tan A;$$

$$\therefore \;\; (1 - \tan A)^2 = \{\sqrt{1+x} \,.\, \tan A - \sqrt{1-x}\}^2;$$

$$\text{Whence,} \;\; 2 \tan A = x \,.\, \{1 - \tan^2 A\} + 2\sqrt{1-x^2} \,.\, \tan A;$$

$$\therefore \;\; 1 = x \,.\, \frac{1 - \tan^2 A}{2 \tan A} + \sqrt{1-x^2},$$

$$\therefore \;\; \sqrt{1-x^2} = 1 - x \,.\, \cot 2A; \ldots (47)$$

$$\therefore \;\; 1 - x^2 = 1 - 2x \,.\, \cot 2A + x^2 \,.\, \cot^2 2A;$$

$$\therefore \;\; x^2 \,.\, (1 + \cot^2 2A) - 2x \,.\, \cot 2A = 0;$$

$$\therefore \;\; x = 0; \;\; \text{or } x = \frac{2 \cot 2A}{1 + \cot^2 2A} = 2 \,.\, \frac{\cot 2A}{\operatorname{cosec}^2 2A} = 2 \sin 2A \,.\, \cos 2A = \sin 4A.$$

(7) *To prove that* $\operatorname{Tan}^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = 45°$.

$$\operatorname{Tan}^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{\frac{1}{3}+\frac{1}{5}}{1 - \frac{1}{3}\cdot\frac{1}{5}}, \;\; (65). \qquad = \tan^{-1}\frac{4}{7},$$

$$\text{So} \;\; \operatorname{Tan}^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \tan^{-1}\frac{\frac{1}{7}+\frac{1}{8}}{1 - \frac{1}{7}\cdot\frac{1}{8}} = \tan^{-1}\frac{3}{11};$$

$$\therefore \;\; \operatorname{Tan}^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8}$$

$$= \tan^{-1}\frac{4}{7} + \tan^{-1}\frac{3}{11} = \tan^{-1}\frac{\frac{4}{7}+\frac{3}{11}}{1 - \frac{4}{7}\cdot\frac{3}{11}} = \tan^{-1}\frac{65}{65} = \tan^{-1} 1 = 45°.$$

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68. The Appendices I, II, III, on the Logarithmic Tables of Numbers and of Goniometrical Ratios, ought to be read before entering on the next Chapter.

CHAPTER IV.

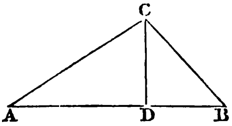
ON THE SOLUTION OF TRIANGLES.

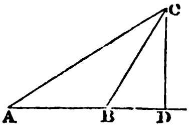
69. A triangle consists of six parts, namely, three sides and three angles. When three of these parts are given, (except they be the three angles), it will be shewn that the other three can, generally, be determined.

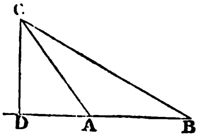
The number of degrees in the angles of a triangle will be designated by the letters A, B, C, which are placed at the angular points of the triangle; and the lengths of the sides respectively opposite to the angles A, B, C, by the letters a, b, c.

70. *The Sines of the Angles of a Triangle are proportional to the Sides respectively opposite to them.*

Let ABC be the triangle. From C draw CD perpendicular to AB, or AB produced either way.







Then $\operatorname{Sin} CAB=\sin CAD=\frac{DC}{CA}$.

(With reference to the third figure, see Art. 26.)

$$\text{Also, } \operatorname{Sin} CBA=\sin CBD=\frac{DC}{CB};$$

$$\therefore \frac{\sin CAB}{\sin CBA}=\frac{CB}{CA};$$

$$\therefore \frac{\operatorname{Sin} A}{\operatorname{Sin} B}=\frac{a}{b}; \qquad \text{or, } \frac{\operatorname{Sin} A}{a}=\frac{\operatorname{Sin} B}{b}.$$

In like manner, if a perpendicular were let fall from B upon the side opposite to it, or that side produced, it might be proved that

$$\frac{\text{Sin}\,A}{a} = \frac{\text{Sin}\,C}{c}.$$

Wherefore, $\frac{\text{Sin}\,A}{a} = \frac{\text{Sin}\,B}{b} = \frac{\text{Sin}\,C}{c}$;—the magnitudes of the lines a, b, c, being represented by the *number* of units of length they respectively contain: for otherwise Sin A and a would not be quantities of the same kind, and consequently no ratio could exist between them.

71. Since (Euclid I. 32), the interior angles of a triangle are together equal to two right angles,

$$A + B + C = 180^\circ.$$

$$\text{Also,} \quad \frac{\text{Sin}\,A}{\text{Sin}\,B} = \frac{a}{b}; \quad \text{and} \quad \frac{\text{Sin}\,B}{\text{Sin}\,C} = \frac{b}{c}.$$

And three of the parts of the triangle being given, the remaining three parts may be determined by these three equations.

One, at least, of the three given parts must be a side, or the *ratios* only are given which a, b, c, bear to each other, and their *magnitudes* cannot be determined, because there are but *two* equations given,

$$\left(\text{namely,} \quad \frac{\text{Sin}\,A}{\text{Sin}\,B} = \frac{a}{b}, \quad \text{and} \quad \frac{\text{Sin}\,B}{\text{Sin}\,C} = \frac{b}{c}\right),$$

for determining the *three* unknown quantities a, b, c.

And this is also apparent from the consideration that an indefinite number of equiangular triangles, of all possible degrees of magnitude, can be formed by drawing lines parallel to the sides of a given triangle.

72. There is however one case, commonly called "*the ambiguous case*", in which the equations of the last article are not sufficient to determine the triangle when three of the parts are given.

If two Sides be given, and an Angle opposite to one of them (a, b, A), *the triangle can be determined only when the side opposite to the given angle is the greater of the two given sides;* i.e. *when* a *is greater than* b.

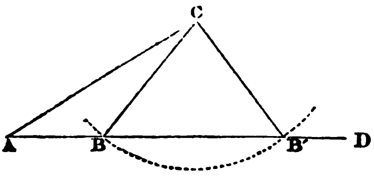
The equations of the last article are

$$\text{(i)}\ B+C=180^\circ-A,\qquad \text{(ii)}\ \operatorname{Sin} B=\frac{b}{a}\cdot\sin A,\qquad \text{(iii)}\ c=b\cdot\frac{\sin C}{\sin B}.$$

If B can be determined from (ii), C and c are known from (i) and (iii), and the triangle is determined. But since the sine of an angle is equal to the sine of its supplement, there are *two* values of the angle B which satisfy (ii), the one greater and the other less than 90°.

(1) Let a be greater than b; $\therefore A>B$. Eucl. I. 18.
Now B cannot be greater than 90°; for if it could, then $A+B$ would be greater than 180°, which is impossible, (Eucl. I. 17.): $\therefore B<90^\circ$, and the *lesser* angle which satisfies (ii) is to be taken for the value of B.

(2) Let a be less than b; $\therefore A<B$.
In this case, since the only limitations, $B+A<180^\circ$, and $B>A$, may be fulfilled whether B be greater or less than 90°, it is impossible to determine from (ii) whether B be $>$ or $<90^\circ$.



Thus if $CB=CB'$, it is evident that *both* the triangles ABC, $AB'C$ have a, b, A of the same values; also, in this case,

$\angle A$ is less than the exterior angle CBB', or the exterior angle $CB'D$, i.e. A is less than $CB'A$, or CBA; and also $a<b$. Which agrees with what has been asserted above.

73. *To find the Cosine of an angle of a Triangle in terms of the Sides.* (Figs. Art. 70).

Let ABC be a triangle, and from C draw CD perpendicular to AB, or AB produced either way.

Then, figs. 1, 2 of Art. 70, $CB^2 = AC^2 + AB^2 - 2AB\,.\,AD$, Eucl. II. 13.

fig. 3, $CB^2 = AC^2 + AB^2 + 2AB\,.\,AD$, Eucl. I. 12.

$$\text{And}\quad \frac{AD}{AC} = \cos CAD, \quad = \cos CAB \text{ in figs. 1, 2.}$$

$$= -\cos CAB \text{ in fig. 3...(26).}$$

Therefore in each of these cases,

$$CB^2 = AC^2 + AB^2 - 2AB\,.\,AC\,.\cos CAB\,;$$

$$\therefore\ \operatorname{Cos} CAB = \frac{AC^2 + AB^2 - CB^2}{2AB\,.\,AC}\,; \quad \text{Or}\quad \operatorname{Cos} A = \frac{b^2 + c^2 - a^2}{2bc}\,.$$

74. *To shew that* $\operatorname{Cos} \frac{1}{2} \mathrm{A} = \sqrt{\frac{\mathrm{S}\,.\,(\mathrm{S} - \mathrm{a})}{\mathrm{bc}}}$; *where* $\mathrm{S} = \frac{1}{2}(\mathrm{a} + \mathrm{b} + \mathrm{c})$.

$$\text{For}\quad 1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{b^2 + 2bc + c^2 - a^2}{2bc}$$

$$= \frac{(b+c)^2 - a^2}{2bc} = \frac{(b+c+a)\,.\,(b+c-a)}{2bc}\,.$$

Now, $1 + \cos A = 2\cos^2 \frac{1}{2}A$, (39. 2).

And $S - a = \frac{1}{2}(a + b + c) - a = \frac{1}{2}(b + c - a)\,;$

$$\therefore\ 2\cos^2 \tfrac{1}{2}A = \frac{(b+c+a)\,.\,(b+c-a)}{2bc} = \frac{2S\,.\,2(S-a)}{2bc}\,;$$

$$\therefore\ \operatorname{Cos} \tfrac{1}{2}A = \sqrt{\frac{S\,.\,(S-a)}{bc}}\,.$$

75. *To shew that* $\operatorname{Sin} \frac{1}{2}\mathrm{A} = \sqrt{\frac{(\mathrm{S}-\mathrm{b})\,.\,(\mathrm{S}-\mathrm{c})}{\mathrm{bc}}}$.

By Art. 39. (3),

$$2\sin^2 \tfrac{1}{2}A = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - (b-c)^2}{2bc}$$

$$= \frac{(a+c-b)\,.\,(a+b-c)}{2bc}\,.$$

$$\text{Now}\quad S-b=\tfrac{1}{2}(a+b+c)-b=\tfrac{1}{2}(a+c-b),$$

$$\text{and so}\quad S-c=\tfrac{1}{2}(a+b-c);$$

$$\therefore\ 2\sin^2\tfrac{1}{2}A=\frac{2(S-b)\,.\,2(S-c)}{2bc};$$

$$\therefore\ \operatorname{Sin}\tfrac{1}{2}A=\sqrt{\frac{(S-b)\,.\,(S-c)}{bc}}.$$

The *positive* sign of the root must be taken both here and in the last Article; because A, being an angle of a triangle, is less than 180°, and therefore Cos $\frac{1}{2}A$ and Sin $\frac{1}{2}A$ are necessarily positive quantities.

76. $\operatorname{Sin}^2 A=1-\cos^2 A=(1+\cos A)\,.\,(1-\cos A)$

$$=\left\{1+\frac{b^2+c^2-a^2}{2bc}\right\}.\left\{1-\frac{b^2+c^2-a^2}{2bc}\right\}$$

$$=\frac{1}{4b^2c^2}\,.\,(b+c-a)\,.\,(b+c+a)\,.\,(a+b-c)\,.\,(a+c-b)$$

$$=\frac{1}{4b^2c^2}\,.\,2S\,.\,2(S-a)\,.\,2(S-b)\,.\,2(S-c)$$

$$=\frac{4}{b^2c^2}\,.\,S\,.\,(S-a)\,.\,(S-b)\,.\,(S-c);$$

$$\therefore\ \operatorname{Sin}A=\frac{2}{bc}\sqrt{\{S\,.\,(S-a)\,.\,(S-b)\,.\,(S-c)\}}.$$

$$\text{So,}\quad \operatorname{Tan}\tfrac{1}{2}A=\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A}=\sqrt{\frac{(S-b)\,.\,(S-c)}{S\,.\,(S-a)}}.$$

77. To explain the meaning of the double sign of the second member of the equation

$$\operatorname{Cos}\tfrac{1}{2}A=\pm\sqrt{\frac{S\,.\,(S-a)}{bc}}.$$

Since $\operatorname{Cos}(2n\,.\,180^\circ\pm A)=\cos A$, (25); $\therefore\ \cos(2n\,.\,180^\circ\pm A)=\dfrac{b^2+c^2-a^2}{2bc}$.

Whence it may be proved in the same manner as in (74), that

$$\operatorname{Cos}(n\,.\,180^\circ\pm\tfrac{1}{2}A)=\pm\sqrt{\frac{S\,.\,(S-a)}{bc}}.$$

Now the first member of this equation ought to furnish two sets of angles whose cosines are of the same magnitude but are of different signs.

First. Let n be even, and $= 2m$.

Then $\operatorname{Cos}(n \, . \, 180^\circ \pm \frac{1}{2} A) = \cos(2m \, . \, 180^\circ \pm \frac{1}{2} A) = \cos(m \, . \, 360^\circ \pm \frac{1}{2} A)$; which, by making m equal to 0, 1, 2, 3... successively, gives the series of angles $\pm \frac{1}{2} A$, $360^\circ \pm \frac{1}{2} A$, $2 \times 360^\circ \pm \frac{1}{2} A$, $3 \times 360^\circ \pm \frac{1}{2} A$, ... all the cosines of which are of the same magnitude as that of $\pm \frac{1}{2} A$, or of $+ \frac{1}{2} A$.

Second. Let n be odd, and $= 2m + 1$.

Then $\operatorname{Cos}(n \, . \, 180^\circ \pm \frac{1}{2} A) = \cos\{2m \, . \, 180^\circ + (180^\circ \pm \frac{1}{2} A)\}$; which, by making m equal to 0, 1, 2, 3... successively, gives the series of angles $180^\circ \pm \frac{1}{2} A$, $360^\circ + (180^\circ \pm \quad A)$, $2 \times 360^\circ + (180^\circ \pm \frac{1}{2} A)$, ... all the cosines of which are of the same magnitude as that of $180^\circ \pm A$, which

$$= \cos(180^\circ \pm \tfrac{1}{2} A) = -\cos \tfrac{1}{2} A.$$

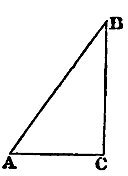
Wherefore the first member of the equation *does* furnish two sets of angles whose cosines are of the same magnitudes but are of different signs.

And in like manner sets of angles may be determined corresponding to each of the signs affecting the expressions which have been obtained as the values of $\operatorname{Sin} \frac{1}{2} A$, $\operatorname{Tan} \frac{1}{2} A$, and $\operatorname{Sin} A$.

I. ON THE SOLUTION OF RIGHT-ANGLED TRIANGLES.

78. *The right angle, a side, and another part being given, to determine the remaining parts.*

Let ABC be a right-angled triangle, C being the right angle.



(1) *Let* c *and* A *be the other given parts.*

Then $\dfrac{b}{c} = \cos A$; and $\dfrac{a}{c} = \sin A$;

$$\therefore \begin{cases} \mathrm{l}_{10} b = \mathrm{l}_{10} c + L \cos A - 10; & \text{which determines } b, \\ \mathrm{l}_{10} a = \mathrm{l}_{10} c + L \sin A - 10; & \text{which determines } a. \end{cases}$$

Also, $\angle B = 90^\circ - A$; which determines B.

In like manner, if B were the given angle, A might be determined.

(2) *Let* A *and* b *be given.*

Then $\frac{c}{b} = \sec A$; and $\frac{a}{b} = \tan A$;

$$\therefore \begin{cases} \mathrm{l}_{10} c = \mathrm{l}_{10} b + L \sec A - 10; \text{ which determines } c, \\ \mathrm{l}_{10} a = \mathrm{l}_{10} b + L \tan A - 10; \text{ which determines } a. \end{cases}$$

Also, $B = 90^\circ - A$; which determines B.

(3) *Let* A *and* a *be given.*

$$\text{Then } \frac{a}{b} = \tan A; \quad \therefore b = \frac{a}{\tan A},$$

$$\text{And } \mathrm{l}_{10} b = \mathrm{l}_{10} a - L \tan A + 10,$$

$$\text{Again, } \frac{a}{c} = \sin A; \quad \therefore c = \frac{a}{\sin A}, \quad \text{or } = a \,.\, \operatorname{cosec} A.$$

$$\text{And } \mathrm{l}_{10} c = \mathrm{l}_{10} a - L \sin A + 10,$$

$$\text{Again, } B = 90^\circ - A.$$

(4) *Let* a, b *be given.*

Then, $\tan A = \frac{a}{b}$; $\therefore L \tan A = \mathrm{l}_{10} a - \mathrm{l}_{10} b + 10$.

Again, $B = 90^\circ - A$.

Again, $\frac{c}{b} = \sec A$; $\therefore \mathrm{l}_{10} c = \mathrm{l}_{10} b + L \sec A - 10$.

The equation $c = \sqrt{(a^2 + b^2)}$ would give c; but the operation of determining c is tedious, particularly if a and b be large numbers.

(5) *Let* c *and* a *be given.*

Then, $\sin A = \frac{a}{c}$; and $\therefore L \sin A = \mathrm{l}_{10} a - \mathrm{l}_{10} c + 10$.

Again, $\frac{b}{c} = \cos A$; $\therefore \mathrm{l}_{10} b = \mathrm{l}_{10} c + L \cos A - 10$.

Or $b^2 = c^2 - a^2 = (c + a)(c - a)$;

$\therefore \mathrm{l}_{10} b = \frac{1}{2} \{\mathrm{l}_{10}(c + a) + \mathrm{l}_{10}(c - a)\}$;

which determines b without previously finding A.

79. **Different methods must be used in different cases to determine the unknown quantities; for what is said in Appendix III. 11. must always be carefully borne in mind, and such a formula selected in each case as may be calculated to give the result with the greatest practical degree of accuracy.**

Thus in the last case, if b be very small compared with a and c, the angle A is nearly a right angle, and the increment of $\sin A$ corresponding to a small given increment α of the angle, [i. e. $\cos A \,.\, \sin\alpha \,.\, \{1 - \tan A \,.\, \tan \frac{1}{2}\alpha\}$, ...(59)], is inconsiderable, and does not, besides, vary nearly as the increment of the angle. In this case therefore the value of A cannot be determined from the Tables with any great degree of accuracy. (App. III. 11.)

The better way of determining A in such a case is first to find the value of b, and then to determine A from its cosine.

$$\text{Thus}\quad \operatorname{Cos} A = \frac{b}{c}; \quad \therefore\ L\cos A = l_{10} b - l_{10} c + 10,$$

$$= l_{10}\sqrt{(c^2 - a^2)} - l_{10} c + 10,$$

$$= \tfrac{1}{2}\,.\,\{l_{10}(c+a) + l_{10}(c-a)\} + 10 - l_{10} c.$$

80. EXAMPLE. *Given* c = 365·1, *and* a = 348·3, *to find* A.

(The Logarithms employed in this Example and others, will generally be found in the three pages of Logarithms subjoined to Appendices I. and II.)

Here, $c + a = 713{\cdot}4$, and $c - a = 16{\cdot}8$.

Performing the operations indicated in the last line of the last Article,

$$\begin{array}{rr}
l_{10}\, 713{\cdot}4 = & 2{\cdot}8533331 \\
l_{10}\, 16{\cdot}8 = & 1{\cdot}2253093 \\
\hline
2\,)\ & 4{\cdot}0786424 \\
\hline
& 2{\cdot}0393212 \\
& 10 \\
\hline
& 12{\cdot}0393212 \\
l_{10}\, 365{\cdot}1 = & 2{\cdot}5624118 \\
\hline
& 9{\cdot}4769094 \\
\text{And}\ \ L\cos 72^{\circ},\ 33' = & 9{\cdot}4769380 \quad \text{See App. III. 9. Ex. 4.} \\
\hline
\text{Difference} = & 286 \\
\hline
\end{array}$$

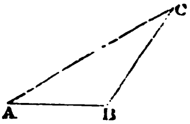
Now Diff. for 1″ is in this case 67·016;

$$\text{and}\ \frac{286}{67{\cdot}016} = 4{\cdot}267 = 4{\cdot}27 \text{ nearly},$$

$$\therefore\ A = 72^{\circ},\ 33',\ 4''{\cdot}27.$$

II. ON THE SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

81. *Let two angles and the side between them be given.* (A, C, b.)



Since $A + B + C = 180^\circ$, $\therefore B = 180^\circ - (A + C)$; which determines B.

Again, $a = b \cdot \dfrac{\sin A}{\sin B}$;

$\therefore \text{l}_{10} a = \text{l}_{10} b + L \sin A - L \sin B$; which determines a.

Again, $c = b \cdot \dfrac{\sin C}{\sin B}$;

$\therefore \text{l}_{10} c = \text{l}_{10} b + L \sin C - L \sin B$; which determines c.

If the value of B be not required; since

$$\text{Sin } B = \sin \{180^\circ - (A + C)\} = \sin (A + C),$$
$$\therefore \text{l}_{10} a = \text{l}_{10} b + L \sin A - L \sin (A + C)$$
$$\text{and } \text{l}_{10} c = \text{l}_{10} b + L \sin C - L \sin (A + C).$$

82. *Let two angles and a side opposite to one of them be given.* (A, C, a.)

Then, $B = 180^\circ - (A + C)$.

Again, $b = a \cdot \dfrac{\sin B}{\sin A}$; $\quad \therefore \text{l}_{10} b = \text{l}_{10} a + L \sin B - L \sin A$,

or $\quad = \text{l}_{10} a + L \sin (A + C) - L \sin A$.

Again, $c = a \cdot \dfrac{\sin C}{\sin A}$; $\quad \therefore \text{l}_{10} c = \text{l}_{10} a + L \sin C - L \sin A$.

83. *Let two sides and the angle included between them be given.* (c, A, b.)

First, to determine B and C.

$$B + C = 180^\circ - A; \qquad \therefore \tfrac{1}{2}(B + C) = \tfrac{1}{2}(180^\circ - A).$$

Also, $\frac{b}{c} = \frac{\sin B}{\sin C}$; $\therefore \frac{\frac{b}{c} - 1}{\frac{b}{c} + 1} = \frac{\frac{\sin B}{\sin C} - 1}{\frac{\sin B}{\sin C} + 1}$;

$$\therefore \frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)}. \quad (52).$$

And $\operatorname{Tan} \frac{1}{2}(B+C) = \tan \frac{1}{2}(180^\circ - A) = \cot \frac{1}{2} A$;

$$\therefore \operatorname{Tan} \frac{1}{2}(B-C) = \frac{b-c}{b+c} . \cot \frac{1}{2} A;$$

$$\therefore L \tan \frac{1}{2}(B-C) = l_{10}(b-c) - l_{10}(b+c) + L \cot \frac{1}{2} A. \quad (2).$$

$\frac{1}{2}(B-C)$ being thus determined, and $\frac{1}{2}(B+C)$ being known,

$$\left. \begin{aligned} B &= \tfrac{1}{2}(B+C) + \tfrac{1}{2}(B-C) \\ \text{and} \quad C &= \tfrac{1}{2}(B+C) - \tfrac{1}{2}(B-C) \end{aligned} \right\} \text{ are known;}$$

And $a = c . \frac{\sin A}{\sin C}$ determines a.

84. *The side* a *can be determined without previously determining* B *and* C. (c, A, b.)

For
$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc . \cos A \\ &= b^2 + c^2 - 2bc . (1 - 2 \sin^2 \tfrac{1}{2} A) \ldots\ldots 39. \ (3). \\ &= (b-c)^2 + 4bc . \sin^2 \tfrac{1}{2} A \\ &= (b-c)^2 . \{1 + \frac{4bc}{(b-c)^2} . \sin^2 \tfrac{1}{2} A\} \\ &= (b-c)^2 . \left\{ 1 + \left(\frac{2\sqrt{bc}}{b-c} . \sin \tfrac{1}{2} A \right)^2 \right\}. \end{aligned}$$

Now $\frac{2\sqrt{bc}}{b-c} . \sin \frac{1}{2} A$ may be of any magnitude and sign, and therefore there is *some* angle of which the tangent is equal to this quantity. Let θ be the angle.

$$\text{Then} \quad \operatorname{Tan} \theta = \frac{2\sqrt{bc}}{b-c} . \sin \tfrac{1}{2} A \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$\text{And}\quad a^2 = (b-c)^2 \,.\, (1+\tan^2\theta) = (b-c)^2 \,.\, \sec^2\theta\,;$$

$$\therefore\ a = (b-c)\,.\,\sec\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$\text{From (1),}\quad L\tan\theta = \mathrm{l}_{10}2 + \tfrac{1}{2}\,\mathrm{l}_{10}b + \tfrac{1}{2}\,\mathrm{l}_{10}c - \mathrm{l}_{10}(b-c) + L\sin\tfrac{1}{2}A,$$

$$= \tfrac{1}{2}(\mathrm{l}_{10}b + \mathrm{l}_{10}c) + \mathrm{l}_{10}2 + L\sin\tfrac{1}{2}A - \mathrm{l}_{10}(b-c)\,;$$

which determines θ.

From (2), $\mathrm{l}_{10}a = \mathrm{l}_{10}(b-c) + L\sec\theta - 10$; which determines a.

85. If $b = c$ nearly, $b - c$ is very small, and $\frac{2\sqrt{bc}}{b-c}\,.\,\sin\frac{1}{2}A$, the value of $\tan\theta$, is very large, (unless $\frac{1}{2}A$, and therefore $\sin\frac{1}{2}A$, be very small). And since the tangents of angles which are nearly right angles are very large, θ in this case is nearly a right angle.

Now if it be required to find from its tangent, an angle which does not consist of a certain number of degrees and minutes exactly, the additional seconds have to be determined on the principle that the increment of the tabular logarithm varies as the increment of the angle. But when the angle is equal to $(2n+1)\,.\,90^\circ$ nearly, this principle does not obtain for the tangent, 62, Cor., App. III. 11; and therefore θ cannot in this case be determined near enough to find a with any great degree of exactness from the equation $a = (b-c)\,.\,\sec\theta$.

When therefore c is nearly equal to b, and A is not a very small angle, the following method will give a with more exactness than the last Article does.

$$a^2 = b^2 + c^2 - 2bc\cos A = b^2 + c^2 - 2bc\,.\,\{2\cos^2\tfrac{1}{2}A - 1\}\ldots\ldots 39.\ (2).$$

$$= (b^2 + c^2 + 2bc) - 4bc\,.\,\cos^2\tfrac{1}{2}A = (b+c)^2\,.\left\{1 - \left(\frac{2\sqrt{bc}}{b+c}\,.\,\cos\tfrac{1}{2}A\right)^2\right\}.$$

Now $(\sqrt{b} - \sqrt{c})^2$, or $b - 2\sqrt{bc} + c$, being a square, is necessarily a positive quantity; therefore the positive part of it is greater than the negative part, or $b + c > 2\sqrt{bc}$; $\therefore\ \frac{2\sqrt{bc}}{b+c}$ is fractional, and *à fortiori* $\frac{2\sqrt{bc}}{b+c}\,.\,\cos\frac{1}{2}A$ is fractional.

Let therefore ϕ be the angle whose sine $= \dfrac{2\sqrt{bc}}{b+c}\,.\,\cos\frac{1}{2}A \ldots\ldots\ldots\ldots (1).$

Then, $a^2 = (b+c)^2\,.\,(1 - \sin^2\phi)$, and $\therefore\ a = (b+c)\,.\,\cos\phi \ldots\ldots\ldots\ldots (2).$

$$\text{From (1),}\quad L\sin\phi = \mathrm{l}_{10}2 + \tfrac{1}{2}\mathrm{l}_{10}b + \tfrac{1}{2}\mathrm{l}_{10}c - \mathrm{l}_{10}(b+c) + L\cos\tfrac{1}{2}A,$$

$$= \tfrac{1}{2}(\mathrm{l}_{10}b + \mathrm{l}_{10}c) + \mathrm{l}_{10}2 + L\cos\tfrac{1}{2}A - \mathrm{l}_{10}(b+c).$$

$$\text{From (2),}\quad \mathrm{l}_{10}a = \mathrm{l}_{10}(b+c) + L\cos\phi - 10:$$

which give ϕ and a respectively.

86. *Let two sides be given and an angle opposite to one of them.* (a, b, A.)

It has been shewn in (72) that with these data the solution is ambiguous unless a be greater than b. But if a be greater than b, then

$$\text{Sin}\, B = \frac{b}{a} \,.\, \sin A, \text{ where } B \text{ is an angle less than } 90^{\circ}.$$

$$\text{Also, } C = 180^{\circ} - (A + B); \text{ and, } c = a \,.\, \frac{\sin C}{\sin A}.$$

87. *Let all the sides be given.* (a, b, c.)

$$\text{Now, } \text{Sin}\, A = \frac{2}{bc} \sqrt{\{S \,.\, (S-a) \,.\, (S-b) \,.\, (S-c)\}},$$

$$\text{Sin}\, \tfrac{1}{2} A = \sqrt{\frac{(S-b) \,.\, (S-c)}{bc}},$$

$$\text{Cos}\, \tfrac{1}{2} A = \sqrt{\frac{S \,.\, (S-a)}{bc}},$$

$${}^{*}\,\text{Tan}\, \tfrac{1}{2} A = \sqrt{\frac{(S-b) \,.\, (S-c)}{S \,.\, (S-a)}}.$$

---

[*] When the three sides of a triangle are given, the angle A is more easily found in practice by dividing the triangle into two right-angled triangles in the following manner, than from the formulæ in the text.

If a perpendicular CD be let fall from the vertex on the base or the base produced, it may be shewn from Euclid II. 12, 13, that

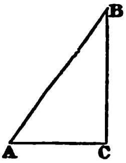
Base
: Sum of the other two sides
:: Difference of the sides
: Difference, or Sum, of the segments of the base;

the fourth term of the proportion being the Difference of the segments of the base, or their Sum, according as the perpendicular cuts the base or the base produced.

And the fourth term of this proportion being found, AD $(= AB \pm BD)$ is known, and thence Cos.A $\left(= \frac{AD}{AC}\right)$ may be determined.

88. Obs. If A be nearly 90°, the first formula of (87) will not give the value of A very exactly, because the increment of $\sin A$ does not in that case vary as the increment of A, and it is also very small; App. III. 11. In this case any one of the last three forms may be used, and the second or the third form must be taken according as $\cos \frac{1}{2} A$ or $\sin \frac{1}{2} A$ is the greater, i. e. as $\frac{1}{2} A$ is less or greater than 45°, (63). The fourth form is applicable in all cases except where $\frac{1}{2} A$ is nearly 90°, (62).

89. Examples. 1. If BC be a perpendicular object standing on a horizontal plane, its height may be determined by measuring in that plane a line AC, which is called *a base*, and observing the angle BAC with a proper instrument.

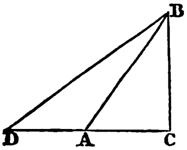


For $BC = AC \,.\, \tan BAC$;

$$\therefore\ l_{10} BC = l_{10} AC + L \tan BAC - 10.$$

2. If it be not practicable to come to the foot of the object, let a base DA be measured, such that the points D, A, C may be in the same straight line; and let the angles BDA, BAC be observed.

Here two angles are given and a side of the $\triangle BAD$. By first determining the side BA the height BC can be found from the right-angled triangle BAC.



$$\text{Thus, } \frac{BA}{AD} = \frac{\sin BDA}{\sin DBA} = \frac{\sin BDA}{\sin (BAC - BDA)}.$$

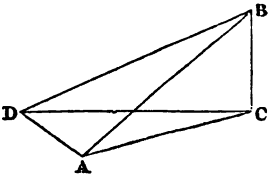
$$\text{And } BC = BA \,.\, \sin BAC = AD \,.\, \frac{\sin BDA \,.\, \sin BAC}{\sin (BAC - BDA)};$$

$$\therefore\ l_{10} BC = l_{10} AD + L \sin BDA + L \sin BAC - L \sin (BAC - BDA) - 10.$$

3. If D be not in the line AC, the height BC can still be determined.

At A let the angles BAC and BAD be observed, and at D the angle BDA.

Then in the $\triangle BDA$, the angles BDA, BAD and the side AD are given. If then BA be determined from these data, BC can be found from the right-angled triangle BAC.



$$\text{Thus, } \frac{BA}{AD} = \frac{\sin BDA}{\sin DBA} = \frac{\sin BDA}{\sin \{180^\circ - (BDA + BAD)\}};$$

$$\therefore\ BA = AD \cdot \frac{\sin BDA}{\sin(BDA + BAD)};$$

$$\text{And}\quad BC = BA . \sin BAC = AD . \frac{\sin BDA . \sin BAC}{\sin(BDA + BAD)};$$

$$l_{10} BC = l_{10} AD + L \sin BDA + L \sin BAC - L \sin(BDA + BAD) - 10.$$

[It is evident that this determination of BC is not affected by the circumstance of D lying out of the horizontal plane which passes through A and C. Hence it follows that if a straight base AD be measured in *any* direction from A, and the angles BAC, BAD, BDA be observed, these data will be sufficient for finding the height of B above the horizontal plane passing through A and C.]

4. Required to find the breadth of a river AD, from observations made from the top of a tower BC of which height is known. (Figure to Ex. 2.)

At B let there be observed the angles of depression of the points D and A below a horizontal line passing through B and parallel to CD. The angles BDC and BAC are equal to these angles of depression; and

$$DA = BA . \frac{\sin DBA}{\sin BDA} = BA . \frac{\sin(BAC - BDA)}{\sin BDA}$$

$$= \frac{BC}{\sin BAC} . \frac{\sin(BAC - BDA)}{\sin BDA}.$$

5. Required the error in height arising from a small given error in an observation of the angle in Example 1.

Let $BC = h,\quad AC = a,\quad \angle BAC = A.$

Let k be the error in height produced by an error α of the observed angle.

$$\text{Then}\quad h = a . \tan A,\quad h + k = a . \tan(A + \alpha);$$

$$\therefore\ k = a . \{\tan(A + \alpha) - \tan A\} = a . \left\{\frac{\sin(A + \alpha)}{\cos(A + \alpha)} - \frac{\sin A}{\cos A}\right\}$$

$$= a . \frac{\sin(A + \alpha) . \cos A - \cos(A + \alpha) . \sin A}{\cos(A + \alpha) . \cos A} = a . \frac{\sin\{(A + \alpha) - A\}}{\cos(A + \alpha) . \cos A}$$

$$= a . \frac{\sin \alpha}{\cos^2 A};\quad \text{since } \cos A = \cos(A + \alpha) \text{ nearly, when } \alpha \text{ is very small, (60).}$$

Cor. Hence it can be determined when the error in height, arising from a small *given* error in the observed angle, will be the least.

$$\text{For } k = a.\frac{\sin \alpha}{\cos^2 A} = \frac{h}{\tan A}\,\frac{\sin \alpha}{\cos^2 A}$$

$$= \frac{h.\sin \alpha}{\sin A.\cos A} = \frac{2h.\sin \alpha}{2\sin A.\cos A}$$

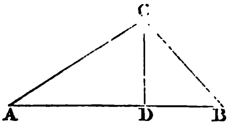
$$= \frac{2h.\sin \alpha}{\sin 2A}.$$

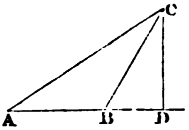
Now h being constant, and α being given, this expression, (which is the error in altitude), will be the least when $\sin 2A$ is the greatest; that is, when $2A = 90^\circ$, or A is 45°.

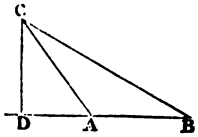
The observer therefore ought to move along the base until $\angle BAC = 45^\circ$, and then by measuring AC he will determine CB (which is in this case equal to AC) with the least chance of error.

90. *To find the Area of a triangle, the sides being given.*

Let ABC be the triangle, and from C draw CD perpendicular to AB, or to AB produced either way.







Then Area of the triangle ABC, being half the rectangle on the same base and between the same parallels,

$$= \tfrac{1}{2} AB.CD = \tfrac{1}{2} AB.AC.\sin CAB$$

$$= \tfrac{1}{2} cb.\frac{2}{bc}.\sqrt{\{S.(S-a).(S-b).(S-c)\}}, \text{ by (76).}$$

$$= \sqrt{\{S.(S-a).(S-b).(S-c)\}}.$$

91. *The Area of the triangle also* $= \tfrac{1}{2} a^2.\dfrac{\sin B.\sin C}{\sin (B+C)}$.

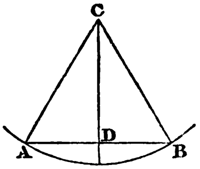
For Area $= \tfrac{1}{2} AB.CD = \tfrac{1}{2} AB.BC.\sin B$

$$= \tfrac{1}{2} BC.\frac{\sin C}{\sin A}.BC.\sin B = \tfrac{1}{2} a^2.\frac{\sin B.\sin C}{(\sin B + C)};$$

since $\text{Sin } A = \sin\{180^\circ - (B+C)\} = (\sin B + C)$.

92. *To find the Radii of the circles described within and about a regular polygon of any number of sides.*

Let AB be a side of a regular polygon of n sides.



Since the polygon is regular, it may have a circle described in and about it; and each of the sides will subtend the same angle at the common centre C of these circles.

Draw CD perpendicular to AB.

Then $AD = DB$, and CD is the radius of the inscribed circle.

Let $r = CD$, and $R = CA$.

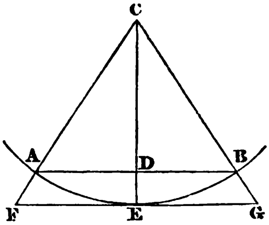
Now the sum of all the angles which the sides subtend at $C = n \times \angle ACB = 360^\circ$;

$$\therefore\ \angle ACB = \frac{360^\circ}{n};\quad \therefore\ \angle ACD = \tfrac{1}{2}\angle ACB = \frac{180^\circ}{n};\ \text{ and } \frac{AD}{CD} = \tan ACD;$$

$$\therefore\ r, = CD, = AD\,.\cot ACD = \tfrac{1}{2}AB\,.\cot\frac{180^\circ}{n}\ \dots\dots\dots\dots\dots\dots\dots\ (1).$$

$$\text{Again, }\ R = AC = \frac{AD}{\sin ACD} = \frac{\tfrac{1}{2}AB}{\sin\frac{180^\circ}{n}} = \tfrac{1}{2}AB\,.\operatorname{cosec}\frac{180^\circ}{n}\ \dots\dots\dots\dots\dots (2).$$

93. *To find the Area of a regular polygon of any number of sides which is described within or about a circle of given radius.*



Let AEB be an arc of the circle whose centre is C; AB a side of the inscribed regular polygon of n sides; CE at right angles to AB, and therefore bisecting it; FG a tangent through E; then FG is a side of the circumscribed polygon of n sides. Let $CA = r$.

$$AB = 2AD = 2\,.\,r\,.\sin ACD = 2r\,.\sin\frac{360^\circ}{2n} = 2r\,.\sin\frac{180^\circ}{n}.$$

$$FG = 2FE = 2r\,.\tan\frac{180^\circ}{n}.$$

$$A = \text{area of inscribed polygon} = n\,.\triangle CAB$$

$$= n\,.\frac{CA\,.\,CB\,.\sin ACB}{2} = \frac{n}{2}\,.\,r^2.\sin\frac{360^\circ}{n}.$$

$$A' = \text{area of circumscribed polygon} = n\,.\triangle CFG$$

$$= n\,.\,CE\,.\,FE = n\,.\,r^2.\tan\frac{180^\circ}{n}.$$

Cor. These areas may be thus compared;

$$\frac{\text{Area of inscribed polygon}}{\text{Area of circumscribed polygon}} = \frac{\triangle CAD}{\triangle CFE} = \frac{CD^2}{CE^2}, \text{ since the } \triangle\text{s are similar,}$$

$$= \left(\frac{CD}{CA}\right)^2 = \cos^2\frac{180^\circ}{n}.$$

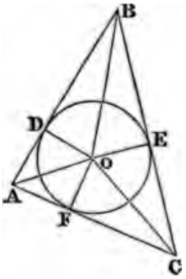
94. *To find the Area of a regular polygon of* n *sides in terms of a side of the polygon.* (Fig. Art. 92.)

AB being a side of the polygon,

$$\text{Area} = n\,.\,\triangle CAB = n\,.\,\tfrac{1}{2}AB\,.\,DC = \tfrac{1}{2}n\,.\,AB\,.\,AD\,.\cot ACD$$

$$= \tfrac{1}{2}n\,.\,AB\,.\,\tfrac{1}{2}AB\,.\cot\frac{180^\circ}{n} = \tfrac{1}{4}n\,.(AB)^2\,.\cot\frac{180^\circ}{n}.$$

95. *To find the Radii of the circles described in and about a triangle of which the sides are given.*



Let the lines bisecting the angles A and B meet in O, and from O draw OD, OE, OF perpendiculars to the sides.

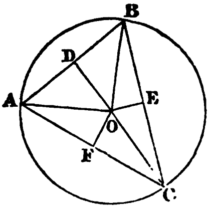
Then, Euclid IV. 4, O is the centre of the inscribed circle; and its radius $r = OD = OE = OF$.

Now, Area of $\triangle ABC = \triangle AOB + \triangle BOC + \triangle COA$;

$$\therefore \sqrt{\{S\,.\,(S-a)\,.\,(S-b)\,.\,(S-c)\}}$$

$$= r\,.\,\tfrac{1}{2}c + r\,.\,\tfrac{1}{2}a + r\,.\,\tfrac{1}{2}b = r\,.\,S;$$

$$\therefore r = \sqrt{\frac{(S-a)\,.\,(S-b)\,.\,(S-c)}{S}} \quad\text{..........................}(1).$$



Again; bisect the sides of the triangle in D, E, F, and draw perpendiculars meeting in a point O which is the centre of the circumscribed circle; R, its radius, $= OA = OB = OC$, Euclid IV. 5.

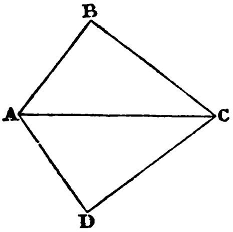
And $\therefore A = \tfrac{1}{2}BOC$, Euclid III. 20;

$$\therefore \sin A = \sin\tfrac{1}{2}BOC = \sin BOE = \frac{BE}{BO} = \frac{\tfrac{1}{2}a}{R};$$

$$\therefore, \text{ by (76), } \frac{2}{bc}\sqrt{\{S\,.\,(S-a)\,.\,(S-b)\,.\,(S-c)\}} = \frac{\tfrac{1}{2}a}{R};$$

$$\therefore R = \frac{abc}{4\sqrt{\{S\,.\,(S-a)\,.\,(S-b)\,.\,(S-c)\}}} \quad\text{..............................}(2).$$

96. *To find the Area of a quadrilateral figure of which the opposite angles are supplements to each other.*



Let $ABCD$ be the quadrilateral.

Let $AB=a$, $BC=b$, $CD=c$, $DA=d$; Join AC.

$$\text{Then Area } ABCD = \triangle ABC + \triangle ADC$$
$$= \tfrac{1}{2}ab \,.\, \sin B + \tfrac{1}{2}cd \,.\, \sin D$$
$$= \tfrac{1}{2}(ab+cd) \,.\, \sin B,$$

since $\text{Sin}\, D = \sin(180^\circ - D) = \sin B$.

Now from $\triangle ABC$, $a^2+b^2-AC^2=2ab \,.\, \cos B$;

and from $\triangle ACD$, $c^2+d^2-AC^2=2cd \,.\, \cos D=-2cd \,.\, \cos B$;

$\therefore$, by subtracting, $a^2+b^2-c^2-d^2=2(ab+cd) \,.\, \cos B$;

$$\therefore \text{Sin}^2 B = 1-\cos^2 B = 1-\left\{\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}\right\}^2$$
$$=\frac{4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2}{4(ab+cd)^2}.$$

And $(\text{area } ABCD)^2 = \frac{1}{4}(ab+cd)^2 . \sin^2 B$;

$$\therefore \quad = \frac{1}{16} . \{4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2\}$$
$$= \frac{1}{16} . \{2(ab+cd)+(a^2+b^2-c^2-d^2)\} . \{2(ab+cd)-(a^2+b^2-c^2-d^2)\}$$
$$= \frac{1}{16} . \{(a+b)^2-(c-d)^2\} . \{(c+d)^2-(a-b)^2\}$$
$$= \frac{1}{16} . (a+b+c-d) . (a+b+d-c) . (c+d+a-b) . (c+d+b-a),$$

and if $S = \frac{1}{2}(a+b+c+d)$,

$$\text{Area } ABCD = \sqrt{\{(S-a) . (S-b) . (S-c) . (S-d)\}}.$$

97. The determination of the unknown quantity in an equation may often be facilitated by breaking the equation up into two others by means of the Goniometrical Ratios. This artifice has been employed in Arts. 84, 85, and the following are additional examples of its use.

EXAMPLES. (1) *Having given* a, *α*, *and* δ, *to determine* l *in a form adapted to logarithmic calculation from the equation,*

$$\text{Sin}\, a = \cos l . \cos \delta . \cos \alpha + \sin l . \sin \delta;$$

(*Hymers' Astronomy*, Art. 154.)

The equation may be thus written,

$$\text{Sin}\, a = \sin\delta \,.\, \left(\sin l + \cos l \,.\, \frac{\cos\delta \,.\, \cos\alpha}{\sin\delta}\right).$$

Now since there are tangents of every magnitude and sign, there is *some* angle ϕ, such that

$$\text{Tan}\,\phi, \text{ or } \frac{\sin\phi}{\cos\phi}, = \frac{\cos\delta \,.\, \cos\alpha}{\sin\delta} = \cot\delta \,.\, \cos\alpha \ldots\ldots\ldots\ldots\text{(1)};$$

$$\therefore\ \text{Sin}\, a = \sin\delta \,.\, \left(\sin l + \cos l \,.\, \frac{\sin\phi}{\cos\phi}\right)$$

$$= \frac{\sin\delta}{\cos\phi} \,.\, (\sin l \,.\, \cos\phi + \cos l \,.\, \sin\phi) = \frac{\sin\delta}{\cos\phi} \,.\, \sin(l+\phi),$$

$$\therefore\ \text{Sin}\,(l+\phi) = \frac{\sin a \,.\, \cos\phi}{\sin\delta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(2)}.$$

From (1), $L\tan\phi = L\cot\delta + L\cos a - 10$, which gives ϕ.

(2), $L\sin(l+\phi) = L\sin a + L\cos\phi - L\sin\delta$,

which gives $l+\phi$, and thence l.

(2) *To express* a . cos θ + b . cos $(\theta + a)$ *under the form* A . cos (B + θ).

Let $\rho = a \,.\, \cos\theta + b \,.\, \cos(\theta+\alpha)$

$$= a \,.\, \cos\theta + b \,.\, \cos\alpha \,.\, \cos\theta - b \,.\, \sin\alpha \,.\, \sin\theta$$

$$= a \,.\, \left(1 + \frac{b}{a} \,.\, \cos\alpha\right) \,.\, \cos\theta - b \,.\, \sin\alpha \,.\, \sin\theta.$$

Let ϕ be the angle whose tangent is $\frac{b}{a} \,.\, \cos\alpha$, or, $\text{Tan}\,\phi = \frac{b}{a} \,.\, \cos\alpha \ldots\ldots\ldots\ldots\text{(1)}$.

$$\therefore\ 1 + \frac{b}{a} \,.\, \cos\alpha = 1 + \tan\phi = \frac{\cos\phi + \sin\phi}{\cos\phi} = \sqrt{2} \,.\, \frac{\frac{1}{\sqrt{2}} \,.\, \cos\phi + \frac{1}{\sqrt{2}} \,.\, \sin\phi}{\cos\phi}$$

$$= \sqrt{2} \,.\, \frac{\sin 45^\circ \,.\, \cos\phi + \cos 45^\circ \,.\, \sin\phi}{\cos\phi} = \sqrt{2} \,.\, \frac{\sin(45^\circ+\phi)}{\cos\phi};$$

$$\therefore\ \rho = a \,.\, \sqrt{2} \,.\, \frac{\sin(45^\circ+\phi)}{\cos\phi} \,.\, \left\{\cos\theta - \frac{b \,.\, \sin\alpha \,.\, \cos\phi}{a \,.\, \sqrt{2} \,.\, \sin(45^\circ+\phi)} \,.\, \sin\theta\right\}.$$

$$\text{Let Tan}\,\phi' = \frac{b \,.\, \sin\alpha \,.\, \cos\phi}{a \,.\, \sqrt{2} \,.\, \sin(45^\circ+\phi)}, \ldots\ldots\ldots\ldots\ldots\ldots\text{(2)}.$$

$$\text{Then } \rho = a\sqrt{2} \,.\, \frac{\sin(45^\circ+\phi)}{\cos\phi \,.\, \cos\phi'} \,.\, \cos(\phi'+\theta); \ldots\ldots\ldots\ldots\ldots\text{(3)}.$$

a result of the required form.

(3) *Adapt* $\sqrt{\frac{a-b}{a+b}}+\sqrt{\frac{a+b}{a-b}}$ *to logarithmic computation.*

$$\sqrt{\frac{a-b}{a+b}}+\sqrt{\frac{a+b}{a-b}}=\frac{a-b+a+b}{\sqrt{(a^2-b^2)}}=\frac{2a}{\sqrt{(a^2-b^2)}}=\frac{2}{\sqrt{\left(1-\frac{b^2}{a^2}\right)}}.$$

Now $\frac{b^2}{a^2}$ is necessarily a positive quantity, and it may be of any magnitude; let therefore θ be an angle such that $\text{Tan}^2\theta=\frac{b^2}{a^2}$; .... (1).

$$\therefore \frac{2}{\sqrt{\left(1-\frac{b^2}{a^2}\right)}}=\frac{2}{\sqrt{\left\{1-\left(\frac{\sin\theta}{\cos\theta}\right)^2\right\}}}=\frac{2\cos\theta}{\sqrt{\cos^2\theta-\sin^2\theta}}$$

$=\frac{2\cos\theta}{\cos^{\frac{1}{2}}2\theta}$; an expression of the required form .... (2).

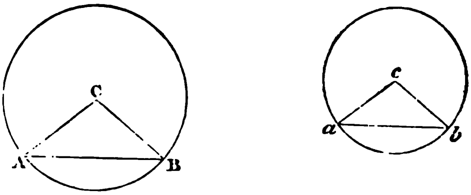
98. **Def.** The angles introduced to assist the solution of an equation, by breaking it up into other two equations or more, are called *Subsidiary Angles.*

CHAPTER V.

ANALYTICAL TRIGONOMETRY.

99. THE *Circumference of a Circle varies as its Radius.*

Let P, p be the perimeters of two regular polygons of n sides each, which are inscribed in two circles whose radii are R, r and circumferences C, c respectively.



Let $X = C - P$, $x = c - p$.

Now when n increases, P and p increase; therefore X and x are variable quantities dependent on the value of n.

Let AB, ab be sides of the polygons; C, c the centres of the circles.

Then $\angle ACB = \dfrac{360^\circ}{n} = \angle acb$; and ACB, acb are similar triangles :

$$\therefore \frac{R}{r} = \frac{AB}{ab} = \frac{n \times AB}{n \times ab} = \frac{P}{p} = \frac{C - X}{c - x};$$

$$\therefore R \times c - R \times x = r \times C - r \times X.$$

Now since $R \times c$, $r \times C$ are constant, and $R \times x$, $r \times X$ are variable quantities, by *Francœur's Pure Mathematics*, Art. 167,

$$R \times c = r \times C, \text{ and } R \times x = r \times X;$$

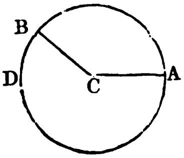
$$\therefore \frac{c}{C} = \frac{r}{R}, \quad \text{or } c \propto r.$$

Cor. 1. Since $c \propto r$; $\therefore \frac{c}{r}$, $\left(\text{or } \frac{\text{the circumference of a circle}}{\text{radius}}\right)$ is a constant quantity.

This constant quantity is always represented by 2π; the approximate value of π, (3·14159...), will be determined hereafter.

Cor. 2. Since $\frac{c}{r} = 2\pi$; $\therefore c = 2\pi r$; or $2\pi r$ represents the circumference of a circle whose radius is r.

100. THE CIRCULAR MEASURE OF AN ANGLE. *If an arc be traced out by a point in the line* CB, *by the revolution of which from the position* CA *an angle* ACB *is described, the angle* ACB *may be properly measured by the ratio* $\frac{\text{arc AB}}{\text{radius AC}}$.



Since in equal circles, (and therefore in the same circle,) angles at the centre have the same ratio to each other as the arcs on which they stand, Euclid VI. 33,

$$\therefore \frac{\angle ACB}{4 \text{ right angles}} = \frac{\text{arc } AB}{\text{whole circumference } ABDA}$$

$$= \frac{AB}{2\pi \,.\, CA}. \quad (99. \text{ Cor. } 2);$$

$$\therefore \angle ACB = \frac{4 \text{ right angles}}{2\pi} \cdot \frac{AB}{CA}.$$

Now, since $\frac{4 \text{ right angles}}{2\pi}$ is a constant quantity, $\frac{AB}{CA}$ increases or decreases in the same ratio as the angle ACB increases or decreases, and therefore $\frac{\text{arc}}{\text{radius}}$ is a proper measure of the magnitude of an angle.

Wherefore an angle may be said to be *equal* to $\frac{\text{the subtending arc}}{\text{radius}}$.

Cor. 1. If $\theta = \frac{\text{arc}}{\text{radius}}$, then $\text{Arc} = r\theta$.

Cor. 2. Since the circular measure of an angle (arc ÷ radius) becomes unity when the arc is equal to the radius, the angle which is subtended by an arc that is equal to the radius is the *Unit of the Circular measurement* of angles.

101. In the preceding Chapters the magnitude of an angle has been measured by the number of times it contains a fixed and definite angle called a degree, and its subdivisions; for several *analytical* investigations, however, the circular measure $\frac{\text{arc}}{\text{radius}}$ is much more convenient. The circular measures of angles will, generally speaking, be denoted by the letters of the Greek Alphabet.

102. *Having given the circular measure* $\left(\frac{\text{arc}}{\text{radius}}\right)$ *of an angle, to determine how many* degrees *the angle contains; and conversely.*

Let θ be the circular measure of the angle which contains A degrees.

Since $\frac{\text{circumference}}{\text{radius}} = 2\pi$, and the circumference subtends four right angles, therefore 2π is the circular measure of four right angles.

Now $\frac{A^0}{360^0} = \frac{\text{given angle}}{4 \text{ right angles}} = \frac{\theta}{2\pi}$;

$$\therefore \ \theta = 2\pi \cdot \frac{A}{360} = \frac{\pi}{180} \cdot A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$= {\cdot}017453 \ldots \times A.$$

$$\text{And} \quad A = 360 \cdot \frac{\theta}{2\pi} = \frac{180}{\pi} \cdot \theta \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$= 57{\cdot}29577 \ \ldots \times \theta.$$

Ex. 1. If $A = 60$,

$$\theta = \frac{60}{180} \cdot \pi = \tfrac{1}{3} \times 3{\cdot}14159 \ldots = 1{\cdot}04719 \ldots\ldots$$

Ex. 2. *Required the number of degrees in the angle which is the unit of measurement when the circular measure is used;* i. e. *the degrees subtended by the arc which is equal to the radius.* (100. Cor. 2.)

$$\text{Here} \qquad \theta = \frac{\text{arc}}{\text{radius}} = 1;$$

$$\therefore A = \frac{180}{\pi} \cdot \theta = \frac{180^0}{3{\cdot}1415926} = 57^0{\cdot}29577$$

$$\begin{array}{r} 60 \\ \hline 17{\cdot}7462 \\ 60 \\ \hline 44{\cdot}772 \\ \hline \end{array}$$

And the degrees, minutes, and seconds required are 57°, 17′, 44″·77.

Cor. 1. The number of *seconds* subtended by an arc which is equal to the radius with which it is described, is $57{\cdot}29577 \times 60 \times 60 = 206264{\cdot}772 = 206265$ nearly.

The number of *minutes* subtended by the same arc is 3438 nearly.

Cor. 2. Since the angle subtended by an arc which is equal to the radius contains a degree 57·29577 times, the unit of measurement when the magnitude of an angle is estimated by the circular measure is 57·29577 times as great as the unit of measurement when the angle is expressed in degrees.

103. Four right angles being represented by 2π, and therefore two right angles by π, if the angle A^0 be represented, according to the circular measure, by θ, it follows from 24, 25, that

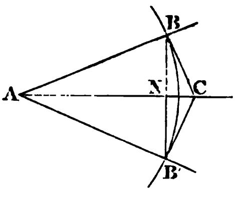
$$\left\{\begin{array}{lll} \text{Sin } \theta = & \sin(2n\pi + \theta), & \text{or } -\sin(2n\pi - \theta). \\ \text{Sin } \theta = & -\sin\{(2n+1)\pi + \theta\}, & \text{or } \ \sin\{(2n+1)\pi - \theta\}. \end{array}\right.$$

$$\left\{\begin{array}{lll} \text{Cos } \theta = & \cos(2n\pi + \theta), & \text{or } \ \cos(2n\pi - \theta). \\ \text{Cos } \theta = & -\cos\{(2n+1)\pi + \theta\}, & \text{or } -\cos\{(2n+1)\pi - \theta\}. \end{array}\right.$$

$$\left\{\begin{array}{lll} \text{Tan } \theta = & \tan(2n\pi + \theta), & \text{or } -\tan(2n\pi - \theta). \\ \text{Tan } \theta = & \tan\{(2n+1)\pi + \theta\}, & \text{or } -\tan\{(2n+1)\pi - \theta\}. \end{array}\right.$$

$$\left\{\begin{array}{lll} \text{Sec } \theta = & \sec\{2n\pi + \theta\}, & \text{or } \ \sec(2n\pi - \theta). \\ \text{Sec } \theta = & -\sec\{(2n+1)\pi + \theta\}, & \text{or } -\sec\{(2n+1)\pi - \theta\}. \end{array}\right.$$

104. *The Circular Measure of an angle less than a right angle is greater than the Sine and less than the Tangent of the angle. Also as the angle decreases, each of the quantities* $\frac{\theta}{\sin\theta}$ *and* $\frac{\theta}{\tan\theta}$ *approaches to unity, and on* θ *vanishing, each does become equal to unity.*



Let θ be an angle less than a right angle, and $= \angle BAC, = \angle CAB'$. From any point C in AC draw CB, CB' perpendicular to AB and AB'; then the triangles CAB, CAB' are similar and equal in every respect. With centre A and radius AB describe a circle; this will pass through B', and BC, $B'C$ will touch the circle in B and B'. Join BB', cutting AC in N.

Then (Legendre's Geometry, IV. 9.), arc BB' is $> BNB'$ and $< BC + CB'$.

$$\therefore \frac{\frac{1}{2} BB'}{AB} > \frac{\frac{1}{2} BNB'}{AB} \text{ and is } < \frac{\frac{1}{2}(BC + CB')}{AB},$$

$$\text{or } \theta > \frac{BN}{AB}, \text{ and is } < \frac{BC}{AB};$$

$$\text{that is, } \theta > \sin\theta, \text{ and is } < \tan\theta.$$

AGAIN; since θ lies between $\sin\theta$ and $\tan\theta$,

$$\frac{\theta}{\sin\theta} \text{ lies between 1 and } \frac{\tan\theta}{\sin\theta}, \left(\text{i. e. } \frac{1}{\cos\theta}\right).$$

But when θ is diminished, $\cos\theta$ is increased; and when θ is diminished indefinitely and vanishes, $\cos\theta$ becomes equal to unity. Wherefore $\frac{\theta}{\sin\theta}$ becomes ultimately equal to unity, and consequently $\frac{\theta}{\tan\theta}$, which $= \frac{\theta}{\sin\theta} \div \frac{1}{\cos\theta}$, likewise ultimately becomes equal to unity. [*Lefebure de Fourcy*].

105. *If θ be an angle less than a right angle,*

$$\operatorname{Sin}\theta < \theta - \tfrac{1}{4}\theta^3.$$

$$\operatorname{Sin}\theta = 2\sin\tfrac{1}{2}\theta \, . \cos\tfrac{1}{2}\theta = \frac{2\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} \, . \cos^2\tfrac{1}{2}\theta;$$

$$\text{but } \frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}, \ (= \tan\tfrac{1}{2}\theta), \text{ is } > \tfrac{1}{2}\theta. \quad (104).$$

$$\therefore\ \text{Sin}\,\theta > 2\,.\,(\tfrac{1}{2}\theta)\,.\cos^2\tfrac{1}{2}\theta$$

$$> \theta\,.\{1-\sin^2\tfrac{1}{2}\theta\}$$

a fortiori,
$$> \theta\{1-(\tfrac{1}{2}\theta)^2\}$$

$$> \theta-\tfrac{1}{4}\theta^3.$$

106. *The Sine of a very small angle is equal to the Circular Measure of the angle, very nearly.*

Since, (105), $\theta-\frac{1}{4}\theta^3<\sin\theta$,

$$\therefore\ \theta-\sin\theta<\tfrac{1}{4}\theta^3.$$

Now if the angle be very small, θ is a very small fraction and $\frac{1}{4}\theta^3$ a still less quantity, so that the circular measure may be written in the place of the sine in any numerical calculation into which such an angle enters without introducing errors of consequence by the substitution.

Cor. If α and β be the circular measures of two very smal angles,

$$\frac{\sin\alpha}{\sin\beta}=\frac{\alpha}{\beta}=\frac{\text{the number of seconds there is in the }\angle\,\alpha}{\text{number of seconds in }\angle\,\beta}.$$

Hence $\dfrac{\sin 2''}{\sin 1''}=\dfrac{2}{1}$, $\dfrac{\sin 3''}{\sin 1''}=\dfrac{3}{1}$;

And, generally, if a be any small number, *but not otherwise,*

$$\frac{\sin n''}{\sin 1''}=\frac{n}{1};$$

$$\therefore\ \text{Sin}\,2''=2\,.\sin 1'',\quad \text{Sin}\,3''=3\,.\sin 1'',\quad \text{Sin}\,n''=n\,.\sin 1''.$$

107. *Required the number of seconds contained in the angle of which θ is the Circular Measure.*

Let a be the required number of seconds.

Now, in the same circle, $\dfrac{\text{Arc subtending } a''}{\text{Arc subtending } 1''}=\dfrac{a}{1}$,

$$\therefore\ a = \frac{\dfrac{\text{arc subtending } a''}{\text{radius}}}{\dfrac{\text{arc subtending } 1''}{\text{radius}}}.$$

But, by hypothesis, $\dfrac{\text{arc subtending } a''}{\text{radius}} = \theta$; and since $1''$ is a very small angle,

$$\therefore\ \frac{\text{arc subtending } 1''}{\text{radius}} = \sin 1'', \text{ nearly; } \quad (105).$$

$$\therefore\ a = \frac{\theta}{\sin 1''},$$

A Result which is often of great practical use.

108. *The Area of a Circle of radius* r *is* πr^2*:*

The Area of a Regular Polygon of n sides inscribed in the Circle is n times the area of a triangle two of whose sides are r, r and the included angle $\frac{2\pi}{n}$;

$$\therefore\ \text{Area of the Polygon} = n\,.\,\frac{r^2}{2}\,.\sin\frac{2\pi}{n} = \pi r^2\,.\,\frac{\sin\frac{2\pi}{n}}{\left(\frac{2\pi}{n}\right)}.$$

Now as the number of the sides increases the Area of the Polygon approaches nearer and nearer to that of the Circle, and when n is infinite becomes identical with it; in which case $\frac{2\pi}{n}$ becomes an indefinitely small angle, and therefore $\sin\frac{2\pi}{n} \div \frac{2\pi}{n}$ becomes 1. (104).

$$\therefore\ \text{Area of the Circle} = \pi r^2.$$

109. DEMOIVRE'S THEOREM. *To shew that for any value of* m,

$$(\cos\theta \pm \sqrt{-1}\sin\theta)^m = \cos m\theta \pm \sqrt{-1}\sin m\theta;$$

the upper signs being taken together, or the lower together.

$$(\text{Cos}\,\theta \pm \sqrt{-1}\sin\theta)\,.\,(\cos\theta \pm \sqrt{-1}\sin\theta)$$
$$= \cos^2\theta - \sin^2\theta \pm \sqrt{-1}\,.\,2\sin\theta\,.\cos\theta.$$

Or, $\quad (\cos\theta \pm \sqrt{-1}\sin\theta)^2 = \cos 2\theta \pm \sqrt{-1}\sin 2\theta.$

Again, $(\cos\theta \pm \sqrt{-1}\,\sin\theta)^2 . (\cos\theta \pm \sqrt{-1}\,\sin\theta)$

$= (\cos 2\theta \pm \sqrt{-1} . \sin 2\theta) . (\cos\theta \pm \sqrt{-1} . \sin\theta)$

$= (\cos 2\theta . \cos\theta - \sin 2\theta . \sin\theta) \pm \sqrt{-1} . (\sin 2\theta . \cos\theta + \cos 2\theta . \sin\theta)$;

$$\therefore\ (\cos\theta \pm \sqrt{-1}\,\sin\theta)^3 = \cos 3\theta \pm \sqrt{-1}\,\sin 3\theta.$$

Suppose this law to hold for m factors, so that

$$(\cos\theta \pm \sqrt{-1}\,\sin\theta)^m = \cos m\theta \pm \sqrt{-1}\,\sin m\theta;$$

then $(\cos\theta \pm \sqrt{-1}\,\sin\theta)^{m+1}$

$= (\cos m\theta \pm \sqrt{-1}\,\sin m\theta) . (\cos\theta \pm \sqrt{-1}\,\sin\theta)$

$= \cos m\theta . \cos\theta - \sin m\theta . \sin\theta \pm \sqrt{-1} . (\sin m\theta . \cos\theta + \cos m\theta . \sin\theta)$

$= \cos (m+1)\theta \pm \sqrt{-1} . \sin (m+1)\theta.$

If therefore the law hold for m factors, it holds for $m+1$ factors; but it has been shewn to hold when $m = 3$, it therefore holds when $m = 4$, and so by successive inductions it may be proved to be true when the index is any positive integer.

Again, $(\cos\theta \pm \sqrt{-1}\,\sin\theta)^{-m} = \left\{\dfrac{1}{\cos\theta \pm \sqrt{-1}\,\sin\theta}\right\}^m$

$$= \left\{\frac{\cos^2\theta + \sin^2\theta}{\cos\theta \pm \sqrt{-1}\,\sin\theta}\right\}^m,$$

$= (\cos\theta \mp \sqrt{-1}\,\sin\theta)^m$, by actual division;

$= \cos m\theta \mp \sqrt{-1}\,\sin m\theta = \cos(-m\theta) \pm \sqrt{-1}\,\sin(-m\theta)$;

which proves the theorem for negative indices.

Again, $(\cos\theta \pm \sqrt{-1}\,\sin\theta)^m = \cos m\theta \pm \sqrt{-1}\,\sin m\theta.$

But $(\cos \frac{m}{n}\theta \pm \sqrt{-1}\,\sin \frac{m}{n}\theta)^n = \cos m\theta \pm \sqrt{-1}\,\sin m\theta$;

$$\therefore \quad (\cos\theta \pm \sqrt{-1}\sin\theta)^m = (\cos\frac{m}{n}\theta \pm \sqrt{-1}\sin\frac{m}{n}\theta)^n;$$

$$\therefore \quad (\cos\theta \pm \sqrt{-1}\sin\theta)^{\frac{m}{n}} = \cos\frac{m}{n}\theta \pm \sqrt{-1}\sin\frac{m}{n}\theta;$$

which proves the theorem for fractional indices.

COR. By the theorem just proved, $(\cos\phi \pm \sqrt{-1}\sin\phi)^m = \cos m\phi \pm \sqrt{-1}\sin m\phi$, m being positive or negative, whole or fractional.

Let $\phi = 2p\pi + \theta$, where p is any integer;

$$\therefore \quad \left.\begin{aligned}\text{Cos}\,\phi &= \cos(2p\pi + \theta) = \cos\theta,\\ \text{Sin}\,\phi &= \sin(2p\pi + \theta) = \sin\theta.\end{aligned}\right\} \quad \text{...(103)}.$$

1. Let the index of $(\cos\theta \pm \sqrt{-1}\sin\theta)$ be integral, as m;

$$\text{Then} \quad \text{Cos}\, m\phi = \cos(2mp\pi + m\theta) = \cos m\theta,$$

$$\text{Sin}\, m\phi = \sin(2mp\pi + m\theta) = \sin m\theta.$$

2. Let the index be fractional, as $\frac{m}{n}$; Then p, being an integer, may be represented by $qn + r$, where q may be 0 or any integer, and r may be 0 or any integer less than n;

$$\therefore \quad \text{Cos}\frac{m}{n}\phi = \cos\frac{m}{n}(2p\pi + \theta) = \cos\frac{m}{n}\{2(qn + r)\pi + \theta\}$$

$$= \cos\{mq\,.\,2\pi + \frac{m}{n}(2r\pi + \theta)\} = \cos\frac{m}{n}(2r\pi + \theta).$$

$$\text{Similarly,} \quad \text{Sin}\frac{m}{n}\phi = \sin\frac{m}{n}(2r\pi + \theta).$$

From these two cases therefore it appears that the theorem might have been thus enunciated;

If the index be an integer,

$$(\cos\theta \pm \sqrt{-1}\sin\theta)^m = \cos m\theta \pm \sqrt{-1}\sin m\theta;$$

If the index be fractional,

$$(\cos\theta \pm \sqrt{-1}\sin\theta)^{\frac{m}{n}} = \cos\frac{m}{n}(2r\pi + \theta) \pm \sqrt{-1}\sin\frac{m}{n}(2r\pi + \theta);$$

where r *may be* 0, *or any integer less than* n.

NOTE. It is to be observed that by giving r all values from 0 to $n - 1$, there will be obtained from the second member of the second equation the n different values of

$$(\cos\theta \pm \sqrt{-1}\sin\theta)^{\frac{m}{n}}.$$

110. *If* $2\cos\theta$ *be represented by* $x+\frac{1}{x}$, *then* $2\sqrt{-1}\,.\sin\theta$ *will be represented by* $x-\frac{1}{x}$, $2\cos m\theta$ *by* $x^m+\frac{1}{x^m}$, *and* $2\sqrt{-1}\sin m\theta$ *by* $x^m-\frac{1}{x^m}$.

$$\text{For } -\sin^2\theta=\cos^2\theta-1=\tfrac{1}{4}\left(x+\frac{1}{x}\right)^2-1=\tfrac{1}{4}\left(x^2-2+\frac{1}{x^2}\right);$$

$$\therefore\ 2\sqrt{-1}\,.\sin\theta=x-\frac{1}{x};$$

And since $2\cos\theta=x+\frac{1}{x}$, $\therefore$ by addition and subtraction,

$$x=\cos\theta+\sqrt{-1}\sin\theta;\quad \text{and } \frac{1}{x}=\cos\theta-\sqrt{-1}\sin\theta;$$

$$\therefore\ \begin{cases} x^m=(\cos\theta+\sqrt{-1}\sin\theta)^m=\cos m\theta+\sqrt{-1}\sin m\theta \\ \dfrac{1}{x^m}=(\cos\theta-\sqrt{-1}\sin\theta)^m=\cos m\theta-\sqrt{-1}\sin m\theta, \end{cases}$$

$$\therefore\ x^m+\frac{1}{x^m}=2\cos m\theta;\quad \text{and } x^m-\frac{1}{x^m}=2\sqrt{-1}\sin m\theta.$$

Cor. By taking the equations of the corollary to (109), the following results are obtained;

(1) The index being an integer,

$$2\cos m\theta = x^m+\frac{1}{x^m};\qquad 2\sqrt{-1}\sin m\theta = x^m-\frac{1}{x^m};$$

(2) The index being a fraction,

$$2\cos\frac{m}{n}(2r\pi+\theta) = x^{\frac{m}{n}}+\frac{1}{x^{\frac{m}{n}}};\qquad 2\sqrt{-1}\sin\frac{m}{n}(2r\pi+\theta) = x^{\frac{m}{n}}-\frac{1}{x^{\frac{m}{n}}};$$

where r may be 0 or any integer less than n.

111. *To express any positive integral power of the Cosine of an angle in terms of the Cosines of the multiples of the angle.*

Let $2\cos\theta = x + \frac{1}{x}$; $\therefore\ 2\cos n\theta = x^n + \frac{1}{x^n}$ ...... (110).

Now

$$(2\cos\theta)^n = \left(x+\frac{1}{x}\right)^n = x^n + n\,.\,x^{n-2} + n\,.\,\frac{n-1}{2}\,x^{n-4} + \ldots + n\,.\,\frac{n-1}{2}\,.\,\frac{1}{x^{n-4}} + n\,.\,\frac{1}{x^{n-2}} + \frac{1}{x^n}\ (1).$$

$$= \left(x^n + \frac{1}{x^n}\right) + n\,.\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \ldots\ldots\ldots\ldots\ldots \qquad (2).$$

1. If n be even; the last term of (2), or the $(\frac{1}{2}n + 1)^{\text{th}}$ term of (1), is

$$n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\,\ldots\,\frac{n-\frac{1}{2}n+1}{\frac{1}{2}n}.$$

2. If n be odd; the sum of the two middle terms of (1), viz.

the $\{\frac{1}{2}(n-1)+1\}^{\text{th}}$ and the $\{\frac{1}{2}(n-1)+2\}^{\text{th}}$,

(which have the same coefficient), is

$$n\,.\,\frac{n-1}{2}\,\ldots\,\frac{n-\frac{1}{2}(n-1)+1}{\frac{1}{2}(n-1)}\,.\left(x+\frac{1}{x}\right).$$

Hence (2) becomes

$$(2\cos\theta)^n = 2\cos n\theta + n\,.\,2\cos(n-2)\,\theta + \ldots$$

$$+\left\{\begin{array}{l} n\,.\,\dfrac{n-1}{2}\,\ldots\,\dfrac{n-\frac{1}{2}n+1}{\frac{1}{2}n};\ \text{when } n \text{ is even.} \\[2ex] n\,.\,\dfrac{n-1}{2}\,\ldots\,\dfrac{n-\frac{1}{2}(n-1)+1}{\frac{1}{2}(n-1)}\,.\,2\cos\theta;\ \text{when } n \text{ is odd.} \end{array}\right.$$

$$\therefore\ 2^{n-1}\,.\cos^n\theta = \cos n\theta + n\,.\cos(n-2)\,\theta + n\,.\,\frac{n-1}{2}\,.\cos(n-4)\,\theta + \ldots$$

the last term of the series being

$$+\tfrac{1}{2}\,.\,n\,.\,\frac{n-1}{2}\,\ldots\,\frac{n-\frac{1}{2}n+1}{\frac{1}{2}n};\ n \text{ being even.}$$

$$+\,n\,.\,\frac{n-1}{2}\,\ldots\,\frac{n-\frac{1}{2}(n-1)+1}{\frac{1}{2}(n-1)}\,.\cos\theta;\ n \text{ being odd.}$$

Cor. Thus, when $n = 2$, $2\cos^2\theta = \cos 2\theta + 1$,

$n = 3$, $4\cos^3\theta = \cos 3\theta + 3\cos\theta$,

$n = 4$, $8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$.

112. *To express any positive integral power of the Sine of an angle in terms of the Sines and Cosines of the multiples of the angle.*

Let $2\sqrt{-1}\sin\theta = x - \frac{1}{x}$; $\quad \therefore\ 2\sqrt{-1}\sin n\theta = x^n - \frac{1}{x^n}$.

And $(2\sqrt{-1}\sin\theta)^n = \left(x - \frac{1}{x}\right)^n = \left\{x + \left(-\frac{1}{x}\right)\right\}^n$

$$= x^n - n \,.\, x^{n-2} + n \,.\, \frac{n-1}{2} \,.\, x^{n-4} - \ldots + n \,.\, \frac{n-1}{2} \,.\, \frac{(-1)^{n-2}}{x^{n-4}} + n \,.\, \frac{(-1)^{n-1}}{x^{n-2}} + \frac{(-1)^n}{x^n} \ldots\ldots (1).$$

$$= \left\{x^n + \left(\frac{-1}{x}\right)^n\right\} - n \,.\, \left\{x^{n-2} + \left(\frac{-1}{x}\right)^{n-2}\right\} + n \,.\, \frac{n-1}{2} \,.\, \left\{x^{n-4} + \left(\frac{-1}{x}\right)^{n-4}\right\} - \ldots (2).$$

1. Let n be even;

$$\text{then } (\sqrt{-1})^n = (\sqrt{-1})^{2 \cdot \frac{n}{2}} = \{(\sqrt{-1})^2\}^{\frac{n}{2}} = (-1)^{\frac{n}{2}};$$

and the middle, or $\left(\frac{n}{2} + 1\right)^{\text{th}}$, term of the expansion (1) is,

$$(-1)^{\frac{n}{2}} \,.\, n \,.\, \frac{n-1}{2} \,\ldots\ldots\ldots\, \frac{n - \frac{1}{2}n + 1}{\frac{1}{2}n}.$$

Wherefore when n is even the series (2) becomes

$$(-1)^{\frac{n}{2}}\, 2^n . \sin^n\theta = \left(x^n + \frac{1}{x^n}\right) - n \,.\, \left(x^{n-2} + \frac{1}{x^{n-2}}\right) + n \,.\, \frac{n-1}{2} \,.\, \left(x^{n-4} + \frac{1}{x^{n-4}}\right) - \ldots$$

$$+ (-1)^{\frac{n}{2}} \,.\, n \,.\, \frac{n-1}{2} \,\ldots\, \frac{n - \frac{1}{2}n + 1}{\frac{1}{2}n}$$

$$= 2\cos n\theta - n \,.\, 2\cos(n-2)\theta + n \,.\, \frac{n-1}{2} \,.\, 2\cos(n-4)\theta - \ldots$$

$$\ldots\ldots + (-1)^{\frac{n}{2}}\, n \,.\, \frac{n-1}{2} \,\ldots\, \frac{n - \frac{1}{2}n + 1}{\frac{1}{2}n};$$

$$\therefore\ (-1)^{\frac{n}{2}}\, 2^{n-1} . \sin^n\theta = \cos n\theta - n \,.\, \cos(n-2)\theta + n \,.\, \frac{n-1}{2} \,.\, \cos(n-4)\theta - \ldots$$

$$\ldots\ldots + (-1)^{\frac{n}{2}}\, \tfrac{1}{2} n \,.\, \frac{n-1}{2} \,\ldots\, \frac{n - \frac{1}{2}n + 1}{\frac{1}{2}n}.$$

2. If n be odd, then $(\sqrt{-1})^n = \sqrt{-1}\,.\,(\sqrt{-1})^{n-1} = (-1)^{\frac{n-1}{2}}.\sqrt{-1}$;

and the last binomial in (2) is

$$(-1)^{\frac{1}{2}(n-1)}.\,n.\frac{n-1}{2}\ \dots\ \frac{n-\frac{1}{2}(n-1)+1}{\frac{1}{2}(n-1)}.\left(x-\frac{1}{x}\right).$$

Wherefore, when n is an odd number, (2) becomes

$$(-1)^{\frac{n-1}{2}}2^n.\sqrt{-1}.\sin^n\theta = \left(x^n-\frac{1}{x^n}\right)-n.\left(x^{n-2}-\frac{1}{x^{n-2}}\right)+n.\frac{n-1}{2}.\left(x^{n-4}-\frac{1}{x^{n-4}}\right)-\dots$$

$$\dots\dots(-1)^{\frac{1}{2}(n-1)}.\,n.\frac{n-1}{2}\ \dots\ \frac{n-\frac{1}{2}(n-1)+1}{\frac{1}{2}(n-1)},$$

$$=2\sqrt{-1}.\sin n\theta - n.2\sqrt{-1}.\sin(n-2)\theta+\dots$$

$$\dots\dots+(-1)^{\frac{n-1}{2}}n.\frac{n-1}{2}\dots\frac{n-\frac{1}{2}(n-1)+1}{\frac{1}{2}(n-1)}.2\sqrt{-1}.\sin\theta\,;$$

$$\therefore\ (-1)^{\frac{n-1}{2}}.2^{n-1}\sin^n\theta = \sin n\theta - n.\sin(n-2)\theta + n.\frac{n-1}{2}.\sin(n-4)\theta+\dots\dots$$

$$\dots\dots+(-1)^{\frac{n-1}{2}}n.\frac{n-1}{2}\dots\frac{n-\frac{1}{2}(n-1)+1}{\frac{1}{2}(n-1)}.\sin\theta.^{*}$$

113. *Having given* Tan θ, *to find* Tan nθ.

$$\text{Cos}\,n\theta+\sqrt{-1}\sin n\theta = (\cos\theta+\sqrt{-1}\sin\theta)^n = (\cos\theta)^n.(1+\sqrt{-1}.\tan\theta)^n$$

$$=(\cos\theta)^n.\left\{1+n.\sqrt{-1}.\tan\theta - n.\frac{n-1}{2}.\tan^2\theta - n.\frac{n-1}{2}.\frac{n-2}{3}.\sqrt{-1}.\tan^3\theta+\dots\right\}\ \dots(1).$$

And equating the possible and impossible parts of this equation,

$$\text{Cos}\,n\theta = \cos^n\theta.\left\{1-n.\frac{n-1}{2}.\tan^2\theta+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.\tan^4\theta-\dots\right\}\ \dots.(2).$$

$$\text{Sin}\,n\theta = \cos^n\theta.\left\{n\tan\theta - n.\frac{n-1}{2}.\frac{n-2}{3}.\tan^3\theta+\dots\right\}\ \dots\dots\dots\dots\dots\dots\dots\dots(3).$$

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[*] Cos $n\theta$ can be expanded in terms of cos θ alone, when n is a positive integer, by an algebraical calculation, but the problem is solved more easily by means of the Differential Calculus. [See Gregory's Examples].

$$\therefore\ \operatorname{Tan} n\theta = \frac{\sin n\theta}{\cos n\theta} = \frac{n \cdot \tan\theta - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \tan^3\theta + \ldots}{1 - n \cdot \frac{n-1}{2} \cdot \tan^2\theta + \ldots}.$$

Cor. If n be a positive integer, the series (1) will terminate, and the last term is $(\sqrt{-1}\tan\theta)^n$.

1. Let n be even; then

$$(\sqrt{-1}\tan\theta)^n = (\sqrt{-1})^{2\cdot\frac{n}{2}}\tan^n\theta = (-1)^{\frac{n}{2}} \cdot \tan^n\theta;$$

therefore the last term is $(-1)^{\frac{n}{2}}\tan^n\theta$, and

$$\operatorname{Tan} n\theta = \frac{n\tan\theta - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}\tan^3\theta + \ldots + (-1)^{\frac{n}{2}-1} n\tan^{n-1}\theta}{1 - n \cdot \frac{n-1}{2}\tan^2\theta + \ldots + (-1)^{\frac{n}{2}}\tan^n\theta}.$$

2. Let n be odd; then

$$(\sqrt{-1}\tan\theta)^n = \sqrt{-1} \cdot (\sqrt{-1})^{n-1} \cdot \tan^n\theta = \sqrt{-1} \cdot (-1)^{\frac{n-1}{2}}\tan^n\theta;$$

therefore the last term is $(-1)^{\frac{n-1}{2}}\tan^n\theta$, and

$$\operatorname{Tan} n\theta = \frac{n \cdot \tan\theta - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}\tan^3\theta + \ldots + (-1)^{\frac{n-1}{2}}\tan^n\theta}{1 - n \cdot \frac{n-1}{2}\tan^2\theta + \ldots + (-1)^{\frac{n-1}{2}} n\tan^{n-1}\theta}.^*$$

114. *To shew that* $\operatorname{Cos} a = 1 - \frac{a^2}{1.2} + \frac{a^4}{1.2.3.4} - \ldots$

and $\operatorname{Sin} a = a - \frac{a^3}{1.2.3} + \frac{a^5}{1.2.3.4.5} - \ldots$

By Art. 113, (2),

$$\operatorname{Cos} n\theta = \cos^n\theta \cdot \left\{1 - n \cdot \frac{n-1}{2} \cdot \tan^2\theta + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \tan^4\theta - \ldots\right\}.$$

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* The sign of $n\tan^{n-1}\theta$ is the same as that of $\tan^n\theta$, because every *odd* term in the expansion (1) has the same sign as the term immediately following it; and when n is odd, $n\tan^{n-1}\theta$ is an odd term, and has therefore the same sign as $\tan^n\theta$.

Let $n\theta = a$, and $\therefore n = \frac{a}{\theta}$,

$$\therefore \text{Cos}\, a = \cos^n\theta \, . \left\{1 - \frac{a}{\theta} \, . \frac{\frac{a}{\theta} - 1}{2} \tan^2\theta + \frac{a}{\theta} \, . \frac{\frac{a}{\theta} - 1}{2} \, . \frac{\frac{a}{\theta} - 2}{3} \, . \frac{\frac{a}{\theta} - 3}{4} \, . \tan^4\theta - \dots\right\}$$

$$= \cos^n\theta \, . \left\{1 - a \, . \frac{a - \theta}{2} \, . \left(\frac{\tan\theta}{\theta}\right)^2 + a \, . \frac{a - \theta}{2} \, . \frac{a - 2\theta}{3} \, . \frac{a - 3\theta}{4} \, . \left(\frac{\tan\theta}{\theta}\right)^4 - \dots\right\}.$$

Now this is true whatever be the value that is given to θ; if then 0 be written for θ, (in which case $\frac{\tan\theta}{\theta} = 1$, and $\cos\theta = 1$, Art. 104.),

$$\therefore \quad \text{Cos}\, a = 1 - \frac{a^2}{1 \, . \, 2} + \frac{a^4}{1 \, . \, 2 \, . \, 3 \, . \, 4} - \dots\dots\dots\dots \qquad (1).$$

By a similar substitution in Art. 113, (3),

$$\text{Sin}\, a = a - \frac{a^3}{1 \, . \, 2 \, . \, 3} + \frac{a^5}{1 \, . \, 2 \, . \, 3 \, . \, 4 \, . \, 5} - \dots\dots\dots\dots \qquad (2).$$

If a be less than $\frac{1}{2}\pi$ (or a right angle), the series (1) and (2) here obtained are immediately convergent.

Note. To arrive at these expressions $\frac{\tan\theta}{\theta}$ has been supposed to become 1 when 0 is written for θ, which is the case only when the angle is referred to the *circular* measure. Hence a, which is equal to $n\theta$, is also referred to that measure in the series (1) and (2).

Cor. If a be an angle so small that a^2 and higher powers of a may be neglected when compared with unity, equation (1) becomes

$$\text{Cos}\, a = 1.$$

If a^2, a^3 be retained, but higher powers of a be neglected, (1) and (2) give,

$$\text{Sin}\, a = a - \frac{a^3}{6}; \qquad \text{Cos}\, a = 1 - \frac{a^2}{2};$$

approximations which are often practically very useful.

115. *If* $\epsilon^{\theta\sqrt{-1}}$ *and* $\epsilon^{-\theta\sqrt{-1}}$ *be expanded in terms of* $\theta\sqrt{-1}$ *and* $-\theta\sqrt{-1}$ *in the same manner as* ϵ^x *is expanded in terms of* x *in the series,*

$$\epsilon^x = 1 + x + \frac{x^2}{1 . 2} + \frac{x^3}{1 . 2 . 3} + \dots \quad \text{(App. I. 18. Cor.)}$$

then,

(1) $\text{Cos}\,\theta = \tfrac{1}{2}.(\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}})$. (2) $\text{Sin}\,\theta = \frac{1}{2\sqrt{-1}}.(\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}})$.

(3) $\text{Tan}\,\theta = \frac{1}{\sqrt{-1}}.\frac{\epsilon^{2\theta\sqrt{-1}}-1}{\epsilon^{2\theta\sqrt{-1}}+1}$.

For, $\epsilon^{\theta\sqrt{-1}} = 1 + \theta.\sqrt{-1} - \frac{\theta^2}{1.2} - \frac{\theta^3}{1.2.3}.\sqrt{-1} + \frac{\theta^4}{1.2.3.4} + \dots$

and $\epsilon^{-\theta\sqrt{-1}} = 1 - \theta.\sqrt{-1} - \frac{\theta^2}{1.2} + \frac{\theta^3}{1.2.3}.\sqrt{-1} + \frac{\theta^4}{1.2.3.4} + \dots$

$$\therefore\ \epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}} = 2.(1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \dots) = 2\cos\theta;\ (114).$$

$$\text{and}\ \ \epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}} = 2.\sqrt{-1}.(\theta - \frac{\theta^3}{1.2.3} + \dots) = 2\sqrt{-1}.\sin\theta;$$

$$\therefore\ \text{Cos}\,\theta = \tfrac{1}{2}(\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}}) \dots\dots\dots\dots (1).$$

$$\text{And,}\ \ \text{Sin}\,\theta = \frac{1}{2\sqrt{-1}}(\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}}) \dots\dots\dots\dots (2).$$

$$\text{Also,}\ \ \text{Tan}\,\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{\sqrt{-1}}.\frac{\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}}}{\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}}};$$

and multiplying the numerator and denominator by $\epsilon^{\theta\sqrt{-1}}$,

$$\text{Tan}\,\theta = \frac{1}{\sqrt{-1}}\ .\ \frac{\epsilon^{2\theta\sqrt{-1}} - 1}{\epsilon^{2\theta\sqrt{-1}} + 1} \dots\dots\dots\dots (3).$$

SERIES TO DETERMINE THE VALUE OF π.

116. GREGORIE'S SERIES. *To prove that*

$$\theta = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \dots$$

$\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}} = 2\cos\theta$; and $\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}} = 2\sqrt{-1}.\sin\theta$;

$$\therefore\ \epsilon^{\theta\sqrt{-1}} = \cos\theta + \sqrt{-1}.\sin\theta = \cos\theta.(1 + \sqrt{-1}\ \tan\theta);$$

$$\therefore\ l_\epsilon\, \epsilon^{\theta\sqrt{-1}} = l_\epsilon \cos\theta + l_\epsilon (1 + \sqrt{-1}\tan\theta)$$

$$\theta\sqrt{-1} = l_\epsilon \cos\theta + \left\{\sqrt{-1}\tan\theta - \frac{(\sqrt{-1}\tan\theta)^2}{2} + \frac{(\sqrt{-1}\tan\theta)^3}{3} - \dots\right\} \text{ (App. I. 14. iii.)}$$

$$= l_\epsilon \cos\theta + \sqrt{-1}\tan\theta + \frac{\tan^2\theta}{2} - \sqrt{-1}\,\frac{\tan^3\theta}{3} - \dots$$

and by equating the impossible parts of this equation,

$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \dots$; which is a convergent series if $\tan\theta$ be not greater than 1; that is, if θ be an angle not greater than half a right angle.

Cor. Tangent of half a right angle, (or $\tan\frac{1}{4}\pi$), $= 1$;

$$\therefore\ \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \dots;$$

$$= 2\left(\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \dots\right),$$

a series not convergent enough to be of much use for calculating the numerical value of π.

117. Euler's Series. *To prove that*

$$\frac{\pi}{4} = \left(\frac{1}{2} - \frac{1}{3.2^3} - \frac{1}{5.2^5} - \dots\right) + \left(\frac{1}{3} - \frac{1}{3.3^3} + \frac{1}{5.3^5} - \dots\right).$$

By (65), $\text{Tan}^{-1}1 - \tan^{-1}\frac{1}{2} = \tan^{-1}\dfrac{1-\frac{1}{2}}{1+\frac{1}{2}} = \tan^{-1}\frac{1}{3}$;

$$\therefore\ \tan^{-1}1 = \tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3},$$

and $\text{Tan}^{-1}\frac{1}{2} = \frac{1}{2} - \frac{1}{3.2^3} + \frac{1}{5.2^5} - \dots$; $\text{Tan}^{-1}\frac{1}{3} = \frac{1}{3} - \frac{1}{3.3^3} + \frac{1}{5.3^5} - \dots$; ...(116).

$$\therefore\ \text{Tan}^{-1}1, \text{ or } \frac{\pi}{4}, = \left(\frac{1}{2} - \frac{1}{3.2^3} + \frac{1}{5.2^5} - \dots\right) + \left(\frac{1}{3} - \frac{1}{3.3^3} + \frac{1}{5.3^5} - \dots\right),$$

a series which converges much more rapidly than that deduced in the Corollary to the last Article.

118. **Machin's Series.** *To prove that*

$$\frac{\pi}{4} = 4\left(\frac{1}{5} - \frac{1}{3.5^3} + \frac{1}{5.5^5} - \ldots\right) - \left(\frac{1}{239} - \frac{1}{3.(239)^3} + \frac{1}{5.(239)^5} - \ldots\right)$$

$$4\tan^{-1}\frac{1}{5} = 2 \times 2\tan^{-1}\frac{1}{5} = 2.\tan^{-1}\frac{\frac{1}{5}+\frac{1}{5}}{1-\frac{1}{5}\cdot\frac{1}{5}} = 2\tan^{-1}\frac{5}{12}$$

$$= \tan^{-1}\frac{\frac{10}{12}}{1-\frac{25}{144}} = \tan^{-1}\frac{120}{119},$$

$$\text{and}\quad \tan^{-1}\frac{120}{119} - \tan^{-1}1 = \tan^{-1}\frac{\frac{120}{119}-1}{1+\frac{120}{119}} = \tan^{-1}\frac{1}{239};$$

$$\text{or}\quad 4\tan^{-1}\frac{1}{5} - \frac{\pi}{4} = \tan^{-1}\frac{1}{239};$$

$$\therefore\ \frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239};$$

$$\therefore\ \frac{\pi}{4} = 4\left(\frac{1}{5} - \frac{1}{3.5^3} + \frac{1}{5.5^5} - \cdots\right) - \left\{\frac{1}{239} - \frac{1}{3.(239)^3} + \frac{1}{5.(239)^5} - \cdots\right\}.$$

In this way it is found that $\pi = 3.141592653589793\ldots\ldots$

Cor. Since $\text{Tan}^{-1}\frac{1}{99} + \tan^{-1}\frac{1}{239} = \tan^{-1}\dfrac{\frac{1}{99}+\frac{1}{239}}{1-\frac{1}{99}\cdot\frac{1}{239}}$

$$= \tan^{-1}\frac{338}{23660} = \tan^{-1}\frac{1}{70},$$

$$\therefore\ \frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99};$$

a series by means of which Mr Rutherford (*Phil. Trans.* 1841) succeeded in determining the value of π to 208 places of figures.

MISCELLANEOUS PROPOSITIONS, SHEWING THE APPLICATION OF THE THEOREMS AND FORMULÆ OF THIS CHAPTER.

119. *Having given* $\operatorname{Tan}\alpha$, $\operatorname{Tan}\beta, \ldots\ldots \operatorname{Tan}\lambda$, *to find* $\operatorname{Tan}(\alpha+\beta+\ldots+\lambda)$.

$$(\operatorname{Cos}\alpha + \sqrt{-1}\sin\alpha).(\cos\beta + \sqrt{-1}\sin\beta)$$

$$= (\cos\alpha.\cos\beta - \sin\alpha.\sin\beta) + \sqrt{-1}.(\cos\alpha.\sin\beta + \cos\beta.\sin\alpha)$$

$$= \cos(\alpha+\beta) + \sqrt{-1}.\sin(\alpha+\beta) \ldots\ldots\ldots\ldots (1).$$

Suppose this law to hold for n factors, so that

$$(\cos\alpha + \sqrt{-1}.\sin\alpha).(\cos\beta + \sqrt{-1}.\sin\beta) \ldots (\cos\kappa + \sqrt{-1}.\sin\kappa)$$

$$= \cos(\alpha+\beta+\ldots+\kappa) + \sqrt{-1}.\sin(\alpha+\beta+\ldots+\kappa);$$

by introducing another factor,

$$(\cos\alpha + \sqrt{-1}.\sin\alpha) \ldots (\cos\kappa + \sqrt{-1}.\sin\kappa).(\cos\lambda + \sqrt{-1}.\sin\lambda)$$

$$= \{\cos(\alpha+\beta+\ldots+\kappa) + \sqrt{-1}.\sin(\alpha+\beta+\ldots+\kappa)\}.(\cos\lambda + \sqrt{-1}.\sin\lambda)$$

$$= \cos(\alpha+\beta+\ldots+\lambda) + \sqrt{-1}.\sin(\alpha+\beta+\ldots+\lambda), \text{ by } (1);$$

and therefore the law holds for $n+1$ factors.

But by actual multiplication the law has been shewn to hold for two factors, it must therefore hold for three factors, and thus by successive inductions it is concluded that it holds generally.

Let S_1 = the sum of the quantities $\operatorname{Tan}\alpha$, $\operatorname{Tan}\beta, \ldots$; S_2 = the sum of the products of every two of them; S_3 = the sum of the products of every three of them; and so on.

Then, $\operatorname{Cos}(\alpha+\beta+\ldots+\lambda) + \sqrt{-1}.\sin(\alpha+\beta+\ldots+\lambda)$

$$= (\cos\alpha + \sqrt{-1}.\sin\alpha).(\cos\beta + \sqrt{-1}.\sin\beta) \ldots (\cos\lambda + \sqrt{-1}.\sin\lambda)$$

$$= \cos\alpha.(1 + \sqrt{-1}.\tan\alpha).\cos\beta.(1 + \sqrt{-1}.\tan\beta) \ldots \cos\lambda.(1 + \sqrt{-1}.\tan\lambda)$$

$$= \cos\alpha.\cos\beta \ldots \cos\lambda.(1 + \sqrt{-1}.S_1 - S_2 - \sqrt{-1}.S_3 + S_4 + \ldots).$$

Wood's Algebra, Art. 271.

Hence, by equating the possible and impossible parts of this equation,

$$\text{Cos}\,(\alpha+\beta+\ldots+\lambda)=\cos\alpha\,.\cos\beta\ldots\cos\lambda\,.(1-S_2+S_4-S_6+\ldots)$$

$$\text{Sin}\,(\alpha+\beta+\ldots+\lambda)=\cos\alpha\,.\cos\beta\ldots\cos\lambda\,.(S_1-S_3+S_5-\ldots);$$

$$\therefore\ \text{Tan}\,(\alpha+\beta+\ldots+\lambda)=\frac{\sin(\alpha+\beta+\ldots+\lambda)}{\cos(\alpha+\beta+\ldots+\lambda)}=\frac{S_1-S_3+S_5-\ldots}{1-S_2+S_4-\ldots}.$$

If there be n of the angles α, β, ... λ, it may be shewn as in (113. Cor.) that

$$\text{Tan}\,(\alpha+\beta+\ldots+\lambda)=\frac{S_1-S_3+S_5-\ldots+(-1)^{\frac{n}{2}-1}S_{n-1}}{1-S_2+S_4-\ldots+(-1)^{\frac{n}{2}}S_n},\ n\text{ even.}$$

$$=\frac{S_1-S_3+S_5-\ldots+(-1)^{\frac{n-1}{2}}S_n}{1-S_2+S_4-\ldots+(-1)^{\frac{n-1}{2}}S_{n-1}},\ n\text{ odd.}$$

120. *If* Sin p = sin P . sin (z + p), *required to expand* p *in terms of* Sin P *and of the Sines of* z *and its multiples.*

Hymers' Astronomy, Art. 247.

Since $\text{Sin}\,p=\frac{1}{2\sqrt{-1}}(\epsilon^{p\sqrt{-1}}-\epsilon^{-p\sqrt{-1}})$. (115), the equation becomes by substituting such values of $\text{Sin}\,p$ and $\text{Sin}\,(z+p)$,

$$\frac{1}{2\sqrt{-1}}.(\epsilon^{p\sqrt{-1}}-\epsilon^{-p\sqrt{-1}})=\sin P.\frac{1}{2\sqrt{-1}}(\epsilon^{(z+p)\sqrt{-1}}-\epsilon^{-(z+p)\sqrt{-1}})$$

and, by multiplying both sides of the equation by $2\sqrt{-1}\,.\,\epsilon^{p\sqrt{-1}}$,

$$\epsilon^{2p\sqrt{-1}}-1=\sin P.(\epsilon^{(z+2p)\sqrt{-1}}-\epsilon^{-z\sqrt{-1}})$$

$$=\sin P.\epsilon^{z\sqrt{-1}}.\epsilon^{2p\sqrt{-1}}-\sin P.\epsilon^{-z\sqrt{-1}};$$

$$\therefore\ \epsilon^{2p\sqrt{-1}}(1-\sin P.\epsilon^{z\sqrt{-1}})=1-\sin P.\epsilon^{-z\sqrt{-1}};$$

$$\therefore\ \epsilon^{2p\sqrt{-1}}=\frac{1-\sin P.\epsilon^{-z\sqrt{-1}}}{1-\sin P.\epsilon^{z\sqrt{-1}}};$$

$$\therefore\ 2p\sqrt{-1}={}_{\epsilon}(1-\sin P.\epsilon^{-z\sqrt{-1}})-\mathrm{l}_{\epsilon}(1-\sin P.\epsilon^{z\sqrt{-1}})$$

$$=-\sin P.\epsilon^{-z\sqrt{-1}}-\tfrac{1}{2}\sin^2P.\epsilon^{-2z\sqrt{-1}}-\tfrac{1}{3}\sin^3P.\epsilon^{-3z\sqrt{-1}}+\ldots$$

$$+\sin P.\epsilon^{z\sqrt{-1}}+\tfrac{1}{2}\sin^2P.\epsilon^{2z\sqrt{-1}}+\tfrac{1}{3}\sin^3P.\epsilon^{3z\sqrt{-1}}+\ldots$$

Appendix I. 14. iii.

$$\therefore\ p = \sin P . \frac{1}{2\sqrt{-1}} . (\epsilon^{x\sqrt{-1}} - \epsilon^{-x\sqrt{-1}}) + \tfrac{1}{2} \sin^2 P . \frac{1}{2\sqrt{-1}} . (\epsilon^{2x\sqrt{-1}} - \epsilon^{-2x\sqrt{-1}}) + \ldots$$

$$= \sin P . \sin x + \tfrac{1}{2} \sin^2 P . \sin 2x + \tfrac{1}{3} \sin^3 P . \sin 3x + \ldots$$

Cor. The number of seconds contained in the angle p, which is expressed by the circular measure, is (107)

$$\frac{p}{\sin 1''} = \frac{\sin P}{\sin 1''} . \sin x + \tfrac{1}{2} . \frac{\sin^2 P}{\sin 1''} . \sin 2x + \tfrac{1}{3} . \frac{\sin^3 P}{\sin 1''} . \sin 3x + \ldots$$

$$\text{or } = \frac{\sin P}{\sin 1''} . \sin x + \frac{\sin^2 P}{\sin 2''} . \sin 2x + \frac{\sin^3 P}{\sin 3''} . \sin 3x + \ldots$$

121. In the same manner if $\tan l' = n . \tan l$, and the tangents be expressed in terms of $l\sqrt{-1}$ and $l'\sqrt{-1}$ by the formula of (115), it may be proved that,

$$l' = l - m . \sin 2l + \tfrac{1}{2} m^2 . \sin 4l - \tfrac{1}{3} m^3 . \sin 6l + \ldots$$

$$\text{where } m = \frac{1-n}{1+n}.$$

Cor. The number of seconds in l', is (107)

$$= \frac{l'}{\sin 1''} = \frac{l}{\sin 1''} - \frac{m}{\sin 1''} . \sin 2l + \tfrac{1}{2} . \frac{m^2}{\sin 1''} . \sin 4l - \ldots$$

$$\text{or } = \frac{l}{\sin 1''} - \frac{m}{\sin 1''} . \sin 2l + \frac{m^2 . \sin 4l}{\sin 2''} - \ldots$$

[The angle l, which is the observed latitude of a place, is read off in degrees, minutes, and seconds from the instrument by which the observation is made; therefore to find the degrees, minutes, and seconds in l', the only computation necessary is to determine the number of seconds in the latter part of the series, viz.

$$- \frac{m}{\sin 1''} . \sin 2l + \frac{m^2}{\sin 2''} . \sin 4l - \ldots].$$

122. *To expand* $\dfrac{1}{1 - e \cos\theta}$ *in a series of the Cosines of* θ *and its multiples;* e *being less than* 1.

By hypothesis, e is less than 1; and since $(1-b)^2$, or $1 - 2b + b^2$, being a square, is necessarily positive, $1 + b^2$ is greater than $2b$. Let therefore

$$e = \frac{2b}{1+b^2};$$

$$\therefore\ 1-e^2=1-\frac{4b^2}{(1+b^2)^2}=\left(\frac{1-b^2}{1+b^2}\right)^2; \qquad \therefore\ \frac{1-b^2}{1+b^2}=\sqrt{1-e^2};$$

Whence, $b^2=\dfrac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}}=\dfrac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\cdot\dfrac{1+\sqrt{1-e^2}}{1+\sqrt{1-e^2}}$;

$$\therefore\ b=\frac{e}{1+\sqrt{1-e^2}}.$$

Let $2\cos\theta=x+\dfrac{1}{x}$;

$$\begin{aligned}\therefore\ 1-e\cos\theta&=1-\frac{b}{1+b^2}\cdot\left(x+\frac{1}{x}\right)\\&=\frac{1}{1+b^2}\cdot\left\{1-b\cdot\left(x+\frac{1}{x}\right)+b^2\right\}\\&=\frac{1}{1+b^2}\cdot(1-bx)\cdot\left(1-\frac{b}{x}\right);\end{aligned}$$

$$\begin{aligned}\therefore\ \frac{1}{1-e\cos\theta}&=(1+b^2)\cdot\frac{1}{1-bx}\cdot\frac{1}{1-\dfrac{b}{x}}\\&=(1+b^2)\cdot\left\{\begin{matrix}(1+bx+b^2x^2+b^3x^3+\ldots)\\ \times\left(1+\dfrac{b}{x}+\dfrac{b^2}{x^2}+\dfrac{b^3}{x^3}+\ldots\right)\end{matrix}\right\}\\&=(1+b^2)\cdot\left\{\begin{aligned}&1+b^2+b^4+\ldots\\&+(b+b^3+b^5+\ldots)\cdot\left(x+\frac{1}{x}\right)\\&+(b^2+b^4+b^6+\ldots)\cdot\left(x^2+\frac{1}{x^2}\right)\\&\ldots\ldots\ldots\ldots\ldots\ldots\\&+(b^n+b^{n+2}+b^{n+4}+\ldots)\cdot\left(x^n+\frac{1}{x^n}\right)\\&+\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\}.\end{aligned}$$

Now $1+b^2+b^4+\ldots=\dfrac{1}{1-b^2}$; $\quad b+b^3+b^5+\ldots=\dfrac{b}{1-b^2}$;

And so $b^n+b^{n+2}+b^{n+4}+\ldots\ldots=\dfrac{b^n}{1-b^2}$; &c.

$$\therefore \quad \frac{1}{1 - e\cos\theta} = (1 + b^2) \cdot \left\{\frac{1}{1 - b^2} + \frac{b}{1 - b^2} \cdot 2\cos\theta + \frac{b^2}{1 - b^2} \cdot 2\cos 2\theta + \ldots\right\}$$

$$= \frac{1 + b^2}{1 - b^2} \cdot \{1 + 2b \cdot \cos\theta + 2b^2 \cdot \cos 2\theta + \ldots\}$$

$$= \frac{1}{\sqrt{1 - e^2}} \cdot \{1 + 2b \cdot \cos\theta + 2b^2 \cdot \cos 2\theta + \ldots .\}.$$

Since e is less than 1, b, or $\frac{e}{1 + \sqrt{1 - e^2}}$, is small, and therefore this series converges rapidly.

Cor. The equation to an ellipse is $r = a \cdot \frac{1 - e^2}{1 - e\cos\theta}$, the focus being the pole and θ being measured from that vertex which is the further from the focus;

$$\therefore \quad r = a \cdot \sqrt{1 - e^2} \cdot \{1 + 2b \cdot \cos\theta + 2b^2 \cdot \cos 2\theta + \ldots\}\ ; \quad \text{where } b = \frac{e}{1 + \sqrt{1 - e^2}}.$$

123. The approximations of 114. Cor. to the values of the sine and the cosine of a very small angle, may often be applied to determine the magnitude of Astronomical Corrections, when they are very small quantities.

Ex. 1. *If* Sin $(\omega - y) = \sin\omega \cdot \cos u$, *where* y *is very small, required an approximate value of* y.

Here, $\sin\omega \cdot \cos y - \cos\omega \cdot \sin y = \sin\omega \cdot \cos u$;

$$\therefore \quad (1 - \tfrac{1}{2}y^2) \cdot \sin\omega - y \cdot \cos\omega = \sin\omega \cdot \cos u\ ;$$

$$\therefore \quad y \cdot \cos\omega + \tfrac{1}{2}y^2 \cdot \sin\omega = \sin\omega - \sin\omega \cdot \cos u$$

$$= \sin\omega \cdot (1 - \cos u) = 2\sin\omega \cdot \sin^2\tfrac{1}{2}u\ ;$$

$$\therefore \quad y \cdot \{1 + \tfrac{1}{2}y \cdot \tan\omega\} = 2\tan\omega \cdot \sin^2\tfrac{1}{2}u\ ;$$

$$\therefore \quad y = \frac{2\tan\omega \cdot \sin^2\frac{1}{2}u}{1 + \frac{1}{2}y \cdot \tan\omega} = 2\tan\omega \cdot \sin^2\tfrac{1}{2}u \cdot \{1 + \tfrac{1}{2}y\tan\omega\}^{-1}$$

$$= 2\tan\omega \cdot \sin^2\tfrac{1}{2}u \cdot \{1 - \tfrac{1}{2}y \cdot \tan\omega + \ldots\}.$$

Now neglecting the second and all the succeeding terms of the expansion, as being small when compared with 1, a first approximation (y_1) to the value of y is obtained,

$$y_1 = 2\tan\omega \cdot \sin^2\tfrac{1}{2}u\ ;$$

And by putting for y in the second member of the equation this its *first* approximate value, a *second* approximation (y_2) is thus got;

$$y_2 = 2 \tan \omega \,.\, \sin^2 \tfrac{1}{2} u \,.\, \{1 - \tan \omega \,.\, \sin^2 \tfrac{1}{2} u \,.\, \tan \omega\}$$

$$= 2 \tan \omega \,.\, \sin^2 \tfrac{1}{2} u \,.\, \{1 - \tan^2 \omega \,.\, \sin^2 \tfrac{1}{2} u\}.$$

The number of seconds contained in y_2, is (107.)

$$\frac{y_2}{\sin 1''} = 2 \,.\, \left(\frac{\tan \omega}{\sin 1''} \,.\, \sin^2 \tfrac{1}{2} u - \frac{\tan^3 \omega}{\sin 1''} \sin^4 \tfrac{1}{2} u\right);$$

and the two terms of the expression having been separately determined by means of logarithms, the number of seconds in y_2 is known.

Hymers' Astronomy, Art. 176.

Ex. 2. In the same manner it may be shewn from

$$\text{Cos}\,(z + y) = \sin n \,.\, \sin z \,.\, \cos m + \cos z \,.\, \cos n,$$

when y and n are very small, that

The number of seconds in y is

$$\frac{y}{\sin 1''} = - \frac{n \,.\, \cos m}{\sin 1''} + \frac{n^2}{\sin 2''} \,.\, \cot z \,.\, \sin^2 m,$$

nearly. The terms of this expression are, as in Ex. 1, determined separately by means of logarithms.

Hymers' Astronomy, Art. 161.

124. The expressions of Arts. 115, 110, have been employed in Arts. 120, 122, to expand certain quantities in the form of a series. The operation can be reversed, as in the following instances.

To find the sum of the series,

(1) $\text{Sin}\, a + \sin 2a + \sin 3a + \ldots \qquad + \sin na.$

(2) $\text{Cos}\, a + \cos 2a + \cos 3a + \ldots \qquad + \cos na.$

(1) Let $2\sqrt{-1} \sin a = x - \dfrac{1}{x}$;

$$\therefore\ 2\sqrt{-1} \sin 2a = x^2 - \frac{1}{x^2} \ldots\ldots\ldots\ldots\ldots (110).$$

$$2\sqrt{-1} \sin 3a = x^3 - \frac{1}{x^3}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$2\sqrt{-1} \sin na = x^n - \frac{1}{x^n};$$

∴ representing the sum of the series (1) by S,

$$2\sqrt{-1}\,S = x + x^2 + x^3 + \ldots + x^n$$

$$-\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \ldots - \frac{1}{x^n}$$

$$= x \cdot \frac{x^n - 1}{x - 1} - \frac{1}{x} \cdot \frac{\frac{1}{x^n} - 1}{\frac{1}{x} - 1} = \frac{x^{n+1} + \frac{1}{x^n} - x - 1}{x - 1}$$

$$= \frac{\left(x^{n+\frac{1}{2}} + \frac{1}{x^{n+\frac{1}{2}}}\right) - \left(x^{\frac{1}{2}} + \frac{1}{x^{\frac{1}{2}}}\right)}{x^{\frac{1}{2}} - \frac{1}{x^{\frac{1}{2}}}} = \frac{2\cos\frac{1}{2}(2n+1)\,a - 2\cos\frac{1}{2}a}{2\sqrt{-1}\,\sin\frac{1}{2}a}$$

$$\therefore\ S = \frac{1}{2\sin\frac{1}{2}a} \cdot \left\{\cos\tfrac{1}{2}a - \cos\tfrac{1}{2}(2n+1)\,a\right\} = \frac{\sin\frac{1}{2}na}{\sin\frac{1}{2}a} \cdot \sin\tfrac{1}{2}(n+1)\,a.$$

(2) Similarly, $2\,(\cos a + \cos 2a + \cos 3a + \ldots + \cos na)$

$$= \frac{\left(x^{\frac{2n+1}{2}} - \frac{1}{x^{\frac{2n+1}{2}}}\right) - \left(x^{\frac{1}{2}} - \frac{1}{x^{\frac{1}{2}}}\right)}{x^{\frac{1}{2}} - \frac{1}{x^{\frac{1}{2}}}} = \frac{\sin\frac{1}{2}(2n+1)\,a - \sin\frac{1}{2}a}{\sin\frac{1}{2}a};$$

$$\therefore\ \text{the series (2)} = \frac{\cos\frac{1}{2}(n+1)\,a \cdot \sin\frac{1}{2}na}{\sin\frac{1}{2}a}.$$

125. The formulæ of Chapters II. and III. may sometimes be employed for the like purpose.

To sum the Series,

(1) $\sin a + \sin(a + b) + \sin(a + 2b) + \ldots \quad + \sin\{a + (n-1)\,b\}$,

(2) $\cos a + \cos(a + b) + \cos(a + 2b) + \ldots \quad + \cos\{a + (n-1)\,b\}$.

(1) $\operatorname{Cos}(a - \tfrac{1}{2}b) - \cos(a + \tfrac{1}{2}b) = 2\sin\tfrac{1}{2}b \,.\, \sin a, \ldots (51, 4).$

so $\cos(a + \tfrac{1}{2}b) - \cos(a + \tfrac{3}{2}b) = 2\sin\tfrac{1}{2}b \,.\, \sin(a + b)$

$$\cos(a + \tfrac{3}{2}b) - \cos(a + \tfrac{5}{2}b) = 2\sin\tfrac{1}{2}b \,.\, \sin(a + 2b)$$

....

$$\cos\{a + \tfrac{1}{2}(2n-3)b\} - \cos\{a + \tfrac{1}{2}(2n-1)b\} = 2\sin\tfrac{1}{2}b \,.\, \sin\{a + (n-1)b\};$$

∴ by addition,

$$2\sin\tfrac{1}{2}b \,.\, [\sin a + \sin(a+b) + \sin(a+2b) + \ldots$$
$$+ \sin\{a + (n-1)b\}] = \cos(a - \tfrac{1}{2}b) - \cos\{a + \tfrac{1}{2}(2n-1)b\};$$

∴ the series (1) $= \dfrac{\sin\frac{1}{2}nb}{\sin\frac{1}{2}b} \,.\, \sin\{a + \tfrac{1}{2}(n-1)b\}.$

(2) In like manner, beginning with $\operatorname{Sin}(a + \tfrac{1}{2}b) - \sin(a - \tfrac{1}{2}b) = 2\sin\tfrac{1}{2}b \,.\, \cos a$ and proceeding as in the last case,

$$\operatorname{Cos} a + \cos(a+b) + \cos(a+2b) + \ldots + \cos\{a + (n-1)b\}$$
$$= \frac{\sin\frac{1}{2}nb}{\sin\frac{1}{2}b} \,.\, \cos\{a + \tfrac{1}{2}(n-1)b\}.$$

Cor. Writing a for b in these results, the results of (124) are immediately obtained.

CHAPTER VI.

ON THE SOLUTION OF EQUATIONS AND THE RESOLUTION OF CERTAIN EXPRESSIONS INTO FACTORS.

126. *To solve a Quadratic Equation.*

I. Let $x^2 + px - q = 0$ be the equation.

$$\text{Then } x = -\left\{\frac{p}{2} \mp \sqrt{\left(\frac{p^2}{4} + q\right)}\right\} = -\sqrt{q}\,.\left\{\frac{p}{2\sqrt{q}} \mp \sqrt{\left(\frac{p^2}{4q} + 1\right)}\right\}.$$

$$\text{Let } \frac{4q}{p^2} = \tan^2\theta \text{ .... } (1).$$

$$\therefore\ x = -\sqrt{q}\,.\left\{\frac{1}{\tan\theta} \mp \frac{\sqrt{(\tan^2\theta + 1)}}{\tan\theta}\right\}$$

$$= -\sqrt{q}\,.\,\frac{1 \mp \sec\theta}{\tan\theta} = -\sqrt{q}\,.\,\frac{\cos\theta \mp 1}{\sin\theta}.$$

$$\text{Now } \frac{\cos\theta - 1}{\sin\theta} = \frac{-2\,.\,\sin^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\,.\,\cos\frac{1}{2}\theta} = -\tan\tfrac{1}{2}\theta,$$

$$\frac{\cos\theta + 1}{\sin\theta} = \frac{2\,.\,\cos^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\,.\,\cos\frac{1}{2}\theta} = \frac{1}{\tan\frac{1}{2}\theta}.$$

If therefore x_1 and x_2 be the two roots of the equation,

$$x_1 = \sqrt{q}\,.\tan\tfrac{1}{2}\theta, \text{ .... } (2); \qquad x_2 = \frac{-\sqrt{q}}{\tan\frac{1}{2}\theta}, \text{ .... } (3).$$

From (1), $L\tan\theta = l_{10}2 + \frac{1}{2}\,.\,l_{10}q - l_{10}p + 10$, which determines θ

(2), $l_{10}x_1 = \frac{1}{2}\,.\,l_{10}q + L\tan\frac{1}{2}\theta - 10$,

(3), $l_{10}(-x_2) = \frac{1}{2}\,.\,l_{10}q - L\tan\frac{1}{2}\theta + 10$.

II. If the equation be $x^2 - px - q = 0$, the roots are the same as those of the preceding equation with their signs changed; for the product of the roots with their signs changed $(-q)$ remains the same, and the sum of the roots with their signs changed (the coefficient of the second term) changes its sign, but continues of the same magnitude. (*Wood's Algebra*, 8th Ed., Art. 281.)

III. Let the equation be $x^2 - px + q = 0$; then

$$x = \frac{p}{2}\cdot\left\{1 \pm \sqrt{\left(1 - \frac{4q}{p^2}\right)}\right\}.$$

If $\frac{4q}{p^2}$ be less than 1, assume $\frac{4q}{p^2} = \sin^2\theta$ ....(1).

Then, x_1 and x_2 being the roots of the equation,

$$x_1 = \tfrac{1}{2}\,p\,.\,(1 + \cos\theta) = p\,.\,\cos^2\tfrac{1}{2}\theta \;................(2).$$

$$x_2 = \tfrac{1}{2}\,p\,.\,(1 - \cos\theta) = p\,.\,\sin^2\tfrac{1}{2}\theta \;................(3).$$

$$\begin{array}{ll}
\text{From (1),} & L\sin\theta = \mathrm{l}_{10}2 + \tfrac{1}{2}\,.\,\mathrm{l}_{10}q - \mathrm{l}_{10}p + 10,\\
\quad(2), & \mathrm{l}_{10}x_1 = \mathrm{l}_{10}p + 2L\cos\tfrac{1}{2}\theta - 20,\\
\quad(3), & \mathrm{l}_{10}x_2 = \mathrm{l}_{10}p + 2L\sin\tfrac{1}{2}\theta - 20.
\end{array}$$

IV. If the equation be $x^2 + px + q = 0$, the roots are the same as those of the last equation with their signs changed. (*Wood's Algebra*, Art. 281.)

In the last two cases, if $\frac{4q}{p^2}$ be > 1, let $\frac{4q}{p^2} = \sec^2\theta$;

then $x = \tfrac{1}{2}p\,.\,\{1 \pm \sqrt{-(\sec^2\theta - 1)}\} = \tfrac{1}{2}p\,.\,\{1 \pm \tan\theta\sqrt{-1}\}$,
and both roots of the equation are impossible.

These solutions may be employed in preference to the common method of solution when p and q are very large numbers.

Ex. *Required the roots of* $x^2 + 365{\cdot}42x - 3469{\cdot}1 = 0$.

By case (I.) $L\tan\theta = \mathrm{l}_{10}2 + \tfrac{1}{2}\,.\,\mathrm{l}_{10}3469{\cdot}1 - \mathrm{l}_{10}365{\cdot}42 + 10$.

By the Tables,

$$\begin{array}{rl}
\mathrm{l}_{10}2 = & {\cdot}3010300\\
\tfrac{1}{2}\,.\,\mathrm{l}_{10}3469{\cdot}1 = \tfrac{1}{2}\times 3{\cdot}5402168 = & 1{\cdot}7701084\\
 & 10\\
\hline
 & 12{\cdot}0711384\\
\text{Subtract } \mathrm{l}_{10}365{\cdot}42 = & 2{\cdot}5627923\\
\hline
 & 9{\cdot}5083461 = L\tan 17^{\circ},\ 52',\ \text{nearly};
\end{array}$$

$$\therefore\ \mathrm{l}_{10}x_1 = \tfrac{1}{2}\,.\,\mathrm{l}_{10}3469{\cdot}1 + L\tan 8^{\circ},\ 56' - 10.$$

$$\begin{array}{rl}
\tfrac{1}{2}\,.\,\mathrm{l}_{10}3469{\cdot}1 = & 1{\cdot}7701084\\
L\tan 8^{\circ},\ 56' = & 9{\cdot}1964302\\
\hline
 & 10{\cdot}9665386\\
\text{Subtract} & 10\\
\hline
 & {\cdot}9665386 = \mathrm{l}_{10}9{\cdot}2584,\ \text{nearly}.
\end{array}$$

Again, $\mathrm{l}_{10}(-x_2) = \frac{1}{2}\,.\,\mathrm{l}_{10}\,3469{\cdot}1 - L\tan 8^\circ,\ 56' + 10.$

$$10 + \tfrac{1}{2}\,.\,\mathrm{l}_{10}\,3469{\cdot}1 = 11{\cdot}7701084$$

$$L\tan 8^\circ,\ 56' = 9{\cdot}1964302$$

$$2{\cdot}5736782 = \mathrm{l}_{10}\,374{\cdot}69, \text{ nearly.}$$

Hence the approximate values of the roots of the proposed equation are

$$9{\cdot}2584, \text{ and } -374{\cdot}69.$$

After x_1 was found, x_2 might have been more easily determined from the equation

$-(x_1 + x_2) = 365{\cdot}42.$ *Wood's Algebra*, Art. 271.

127. *To solve a Cubic Equation.*

Let the equation when transformed, if necessary, to another which wants the second term (*Wood's Algebra*, Art. 284), be $x^3 - qx - r = 0$; if $x = \frac{y}{n}$, this equation becomes $y^3 - qn^2y - rn^3 = 0$.

Now $\text{Cos}^3\phi - \frac{3}{4}\cos\phi - \frac{1}{4}\cos 3\phi = 0$. (55. Cor.)

And this equation is identical with the equation $y^3 - qn^2y - rn^3 = 0$, if

(1) $\text{Cos}\,\phi = y$.

(2) $\frac{3}{4} = qn^2$; and $\therefore n = \frac{1}{2}\sqrt{\frac{3}{q}}$.

(3) $\frac{1}{4}\cos 3\phi = rn^3$; and $\therefore \text{Cos}\,3\phi = 4rn^3 = \frac{r}{2}\sqrt{\frac{27}{q^3}}$.

Let a be the least value of 3ϕ which satisfies the equation (3); then one value of y is $\cos\phi$, or $\cos\frac{1}{3}a$.

Also since, (103), $\text{Cos}\,a = \cos(2m\pi \pm a)$, the two other values of y are contained among the values of $\text{Cos}\,\frac{1}{3}(2m\pi \pm a)$.

Now m, being an integer, must be of one of the forms $3p$, $3p \pm 1$,

$$\text{and Cos}\,\frac{2\,.\,3p\,.\,\pi \pm a}{3} = \cos(2p\pi \pm \tfrac{1}{3}a) = \cos\tfrac{1}{3}a. \quad (103).$$

$$\text{Cos}\,\frac{2(3p \pm 1)\pi \pm a}{3} = \cos\left(2p\pi \pm \frac{2\pi \pm a}{3}\right) = \cos\frac{2\pi \pm a}{3}.$$

Wherefore the three values of y are

$$\text{Cos}\,\tfrac{1}{3}\alpha;\quad \text{Cos}\,\tfrac{1}{3}(2\pi+\alpha);\quad \text{Cos}\,\tfrac{1}{3}(2\pi-\alpha);$$

and the three values of x, or $\frac{y}{n}$, are

$$2\sqrt{\frac{q}{3}}\,.\cos\tfrac{1}{3}\alpha;\quad 2\sqrt{\frac{q}{3}}\,.\cos\tfrac{1}{3}(2\pi+\alpha);\quad 2\sqrt{\frac{q}{3}}\,.\cos\tfrac{1}{3}(2\pi-\alpha).$$

128. *The solution of a cubic equation given in the last Article applies to those cases only where all the roots are possible:* i.e. *to the irreducible case of Cardan's rule.* [*Wood's Algebra,* Art. 331.]

Since Cos α is necessarily less than 1,

$$\therefore\ \frac{r}{2}\sqrt{\frac{27}{q^3}} < 1,\quad \frac{r}{2} < \sqrt{\frac{q^3}{27}},\quad \frac{r^2}{4} < \frac{q^3}{27};$$

which is the irreducible case of Cardan's rule.

129. *To resolve the equation* $x^{2n} - 1 = 0$ *into its quadratic factors,* n *being a positive integer.*

Here $x^{2n} = 1$.

Now $(\cos\theta + \sqrt{-1}\,.\sin\theta)^{2n} = \cos 2n\theta + \sqrt{-1}\,.\sin 2n\theta$,

and if $\theta = \frac{m\pi}{n}$, m and n being integers,

$$\begin{aligned}\left(\cos\frac{m\pi}{n} + \sqrt{-1}\,.\sin\frac{m\pi}{n}\right)^{2n} &= \cos 2m\pi + \sqrt{-1}\,.\sin 2m\pi\\ &= 1,\quad \text{since } \sin 2m\pi = 0;\\ &= x^{2n};\end{aligned}$$

$$\therefore\ x = \cos\frac{m\pi}{n} + \sqrt{-1}\,.\sin\frac{m\pi}{n}.$$

Now m, being an integer, must be of the form $p\,.\,2n + r$, where p is 0, or any integer, and r is 0, or any integer less than $2n$.

$$\begin{aligned}\text{Wherefore,}\quad x &= \cos\frac{(p\,.\,2n+r)\pi}{n} + \sqrt{-1}\,.\sin\frac{(p\,.\,2n+r)\pi}{n}\\ &= \cos\left(2p\pi + \frac{r\pi}{n}\right) + \sqrt{-1}\,.\sin\left(2p\pi + \frac{r\pi}{n}\right)\\ &= \cos\frac{r\pi}{n} + \sqrt{-1}\,.\sin\frac{r\pi}{n}\ \ldots\ldots(103).\end{aligned}$$

And for r writing 0, 1, 2, ... $2n-1$, successively in

$$x = \cos\frac{r\pi}{n} + \sqrt{-1}\,.\,\sin\frac{r\pi}{n};$$

(1) If $r = 0$, $\qquad x = 1$,

(2) $\qquad r = 1$, $\qquad x = \cos\frac{\pi}{n} + \sqrt{-1}\,.\,\sin\frac{\pi}{n}$,

(3) $\qquad r = 2$, $\qquad x = \cos\frac{2\pi}{n} + \sqrt{-1}\,.\,\sin\frac{2\pi}{n}$

...... $\qquad$ ....

(n + 1) $\quad r = n$, $\qquad x = -1$,

...... $\qquad$ ....

(2n − 1) $\quad r = 2n-2$, $\quad x = \cos\frac{2n-2}{n}\pi + \sqrt{-1}\,.\,\sin\frac{2n-2}{n}\pi$

$$= \cos\frac{2\pi}{n} - \sqrt{-1}\,.\,\sin\frac{2\pi}{n}, \quad (103).$$

(2n) $\quad r = 2n-1$, $\quad x = \cos\frac{2n-1}{n}\pi + \sqrt{-1}\,.\,\sin\frac{2n-1}{n}\pi$

$$= \cos\frac{\pi}{n} - \sqrt{-1}\,.\,\sin\frac{\pi}{n}.$$

Now the equations (1) and (n + 1) give the quadratic factor $(x-1).(x+1) = x^2 - 1$; the equations (2) and (2n) give

$$\left(x - \cos\frac{\pi}{n} - \sqrt{-1}\,.\,\sin\frac{\pi}{n}\right).\left(x - \cos\frac{\pi}{n} + \sqrt{-1}\,.\,\sin\frac{\pi}{n}\right) = x^2 - 2x\cos\frac{\pi}{n} + 1;$$

and so on;

Wherefore, $x^{2n} - 1 = (x^2 - 1)\,.\left\{x^2 - 2x\cos\frac{\pi}{n} + 1\right\} \ldots \left\{x^2 - 2x\cos\frac{n-1}{n}\pi + 1\right\}.$

130. *To resolve the equation* $x^{2n} - 1 = 0$ *into its quadratic factors,* n *being a positive integer.*

$$x^{2n} + 1 = 0; \quad \therefore\ x^{2n} = -1.$$

Now $(\cos\theta + \sqrt{-1}\,.\,\sin\theta)^{2n} = \cos 2n\theta + \sqrt{-1}\,.\,\sin 2n\theta$. And making $2n\theta = (2m+1)\pi$, and proceeding as in the last Article,

$$x = \cos\frac{2m+1}{2n}\pi + \sqrt{-1}\,.\,\sin\frac{2m+1}{2n}\pi.$$

Also, assuming $m = p \,.\, 2 + r$, where p is 0, or any integer, and r is 0, or any integer less than $2n$, as before,

$$x = \cos \frac{2r+1}{2n} \pi + \sqrt{-1} \,.\, \sin \frac{2r+1}{2n} \pi ;$$

by means of which it may be proved that

$$x^{2n} + 1 = \left\{ x^2 - 2x \cos \frac{\pi}{2n} + 1 \right\} . \left\{ x^2 - 2x \cos \frac{3\pi}{2n} + 1 \right\} \dots$$

$$\dots \left\{ x^2 - 2 \,.\, x \cos \frac{2n-1}{2n} \pi + 1 \right\} .$$

131. *To resolve the equation* $x^{2n+1} - 1 = 0$ *into its quadratic factors,* n *being a positive integer.*

As in the last two Articles,

$$(\cos\theta + \sqrt{-1} \,.\, \sin\theta)^{2n+1} = \cos(2n+1)\theta + \sqrt{-1} \,.\, \sin(2n+1)\theta$$

$$= 1, \text{ if } (2n+1)\theta = 2m\pi, \text{ or } \theta = \frac{2m\pi}{2n+1},$$

$$= x^{2n+1};$$

$$\therefore \quad x = \cos \frac{2m}{2n+1} \pi + \sqrt{-1} \,.\, \sin \frac{2m}{2n+1} \pi ;$$

and making $m = p \,.\, (2n+1) + r$, this becomes

$$x = \cos \frac{2r\pi}{2n+1} + \sqrt{-1} \,.\, \sin \frac{2r\pi}{2n+1} .^*$$

Whence,

$$x^{2n+1} - 1 = (x-1) \,.\, \{ x^2 - 2x \,.\, \cos \frac{2\pi}{2n+1} + 1 \} \dots\dots \{ x^2 - 2x \,.\, \cos \frac{2n\pi}{2n+1} + 1 \} .$$

132. *To resolve the equation* $x^{2n+1} + 1 = 0$ *into its factors,* n *being a positive integer.*

Here $x^{2n+1} = -1$,

$$(\cos\theta + \sqrt{-1} \,.\, \sin\theta)^{2n+1} = \cos(2n+1)\theta + \sqrt{-1} \,.\, \sin(2n+1)\theta,$$

$$\text{and let } (2n+1)\theta = (2m+1)\pi, \quad \text{or } \theta = \frac{2m+1}{2n+1} \pi ;$$

---

* Here if $r = 0$, $x = 1$, and one factor is $x - 1$; the values of x corresponding to $r = 1$ and $r = 2n$ form one quadratic factor; those corresponding to $r = 2$ and $r = 2n - 1$ form another; and so on.

$$\therefore\ (\cos\theta + \sqrt{-1}\,.\sin\theta)^{2n+1} = -1 = x^{2n+1};$$

$$\therefore\ x = \cos\frac{2m+1}{2n+1}\pi + \sqrt{-1}\,.\sin\frac{2m+1}{2n+1}\pi;$$

and making $m = p\,.(2n+1) + r$, this becomes

$$x = \cos\frac{2r+1}{2n+1}\pi + \sqrt{-1}\,.\sin\frac{2r+1}{2n+1}\pi.$$

Whence, as before,

$$x^{2n+1} + 1 = \{x+1\}\,.\,\{x^2 - 2x\cos\frac{\pi}{2n+1} + 1\}\ldots\ldots$$

$$\ldots\ldots\{x^2 - 2x\cos\frac{2n-1}{2n+1}\pi + 1\}.^*$$

133. *To resolve* Sin θ *and* Cos θ *into factors.*

The values of θ which satisfy the equation $\sin\theta = 0$ are 0, $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$, ... $\pm n\pi$, ...; n being any integer whatever.

Hence the series which expresses the value of $\sin\theta$ is divisible by θ, $\theta - \pi$, $\theta + \pi$, $\theta - 2\pi$, $\theta + 2\pi$, ...; *Wood's Algebra*, Art. 269.

Let $\therefore \sin\theta = a\,.\,\theta\,.(\theta-\pi)\,.(\theta+\pi)\,.(\theta-2\pi)\,.(\theta+2\pi)\ldots$, where a is some constant quantity whose value is to be determined.

$$\therefore\ \sin\theta = \pm a\,.\,\theta\,.(\pi-\theta)\,.(\pi+\theta)\,.(2\pi-\theta)\,.(2\pi+\theta)\ldots\ldots$$

$$= \pm a\,.\,\theta\,.\,\pi\left(1-\frac{\theta}{\pi}\right).\,\pi\left(1+\frac{\theta}{\pi}\right).\,2\pi\left(1-\frac{\theta}{2\pi}\right).\,2\pi\left(1+\frac{\theta}{2\pi}\right)\ldots\ldots$$

$$= \pm a\,.\,\pi^2\,.\,2^2\pi^2\,.\,3^2\pi^2\ldots\ldots\times\theta\,.\left(1-\frac{\theta^2}{\pi^2}\right).\left(1-\frac{\theta^2}{2^2\pi^2}\right)\ldots\ldots$$

$$\therefore\ \frac{\sin\theta}{\theta} = \pm a\,.\,\pi^2\,.\,2^2\pi^2\,.\,3^2\pi^2\ldots\ldots\times\left(1-\frac{\theta^2}{\pi}\right).\left(1-\frac{\theta^2}{2^2\pi^2}\right)\ldots\ldots$$

---

* Here the values of x corresponding to the values 0 and $2n$ of r form one quadratic factor; those corresponding to the values 1 and $2n-1$ form another; and so on. The value of x when r is n is -1, and $x+1$ is therefore a factor.

now if θ become 0, $\frac{\sin\theta}{\theta}$ becomes 1. (104), and this equation becomes

$$1 = \pm a \,.\, \pi^2 \,.\, 2^2\pi^2 \,.\, 3^2\pi^2 \,.\, \ldots$$

$$\therefore \ \operatorname{Sin}\theta = \theta \,.\left(1 - \frac{\theta^2}{\pi^2}\right).\left(1 - \frac{\theta^2}{2^2\pi^2}\right).\left(1 - \frac{\theta^2}{3^2\pi^2}\right) \ldots\ldots \qquad (1).$$

Again; the values of θ which satisfy the equation $\operatorname{Cos}\theta = 0$ are $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}$, ... $\pm\frac{2n+1}{2}\pi$, ... ; n being any integer whatever.

Hence, as in the former case,

$$\operatorname{Cos}\theta = \pm b \,.\, \frac{\pi^2}{2^2} \,.\, \frac{3^2\pi^2}{2^2} \,.\, \ldots \left(1 - \frac{2^2\theta^2}{\pi^2}\right).\left(1 - \frac{2^2\theta^2}{3^2\pi}\right) \ldots\ldots$$

Now if $\theta = 0$, this equation becomes $1 = \pm b \,.\, \frac{\pi^2}{2^2} \,.\, \frac{3^2\pi^2}{2^2}$ ......

$$\therefore \ \operatorname{Cos}\theta = \left(1 - \frac{2^2\theta^2}{\pi^2}\right).\left(1 - \frac{2^2\theta^2}{3^2\pi^2}\right).\left(1 - \frac{2^2\theta^2}{5^2\pi^2}\right) \ldots\ldots \qquad (2).$$

Cor. If $\theta = \frac{1}{2}\pi$, the expression for $\operatorname{Sin}\theta$ becomes,

$$= \frac{\pi}{2} \,.\left(1 - \frac{1}{2^2}\right).\left(1 - \frac{1}{4^2}\right).\left(1 - \frac{1}{6^2}\right).\left(1 - \frac{1}{8^2}\right) \ldots\ldots$$

$$= \frac{\pi}{2} \,.\, \frac{2^2 - 1}{2^2} \,.\, \frac{4^2 - 1}{4^2} \,.\, \frac{6^2 - 1}{6^2} \,.\, \frac{8^2 - 1}{8^2} \ldots\ldots$$

$$= \frac{\pi}{2} \,.\, \frac{(2-1).(2+1)}{2^2} \,.\, \frac{(4-1).(4+1)}{4^2} \,.\, \frac{(6-1).(6+1)}{6^2} \ldots\ldots$$

$$\therefore \ \pi = 2 \,.\, \frac{2^2}{1.3} \,.\, \frac{4^2}{3.5} \,.\, \frac{6^2}{5.7} \,.\, \frac{8^2}{7.9} \cdots$$

This is Wallis's Theorem for the determination of π; in which the successive factors become more and more nearly equal to 1.

APPENDIX I.

ON THE LOGARITHMS OF NUMBERS, AND THE CONSTRUCTION OF THE LOGARITHMIC TABLES OF NUMBERS.

1. **Def.** If $n = a^x$, x is called the *Logarithm* of the *Number n* to the *Base a*; or *the Logarithm of a Number to a given Base is that power to which the base must be raised to give the number.*

If a logarithmic formula be generally true whatever may be the value of the base, the logarithms of the quantities involved will be written thus, $\log m$, $\log n$; but if the logarithms are calculated to some particular base, (as 10 for instance), they will be written thus, $\log_{10} m$, $\log_{10} n$; or thus, $\mathrm{l}_{10} m$, $\mathrm{l}_{10} n$.

If $n = a^x$, and while a remains the same, successive values be given to n, and the corresponding values of x be registered, the tables so formed are called "*Tables of a System of Logarithms to the Base a.*"

It will hereafter (10), be shewn that a system of logarithms calculated to the base 10 is attended with peculiar advantages; and for this reason the tables of logarithms in common use have that base.

2. Since if $n = a^x$, $\quad x = \mathrm{l}_a n$;

therefore, in all cases, $\quad n = a^x = a^{\mathrm{l}_a n}$.

Cor. 1. If $x = 1$, n becomes $= a$, and $\therefore \mathrm{l}_a a = 1$.

If $x = 0$, a^x (or n) becomes 1; and $\therefore \mathrm{l}_a 1 = 0$.

Cor. 2. If a be the base of any system,

$$\text{since} \quad m = a^{\mathrm{l}_a m}, \quad \text{and } n = a^{\mathrm{l}_a n},$$

$$\therefore \; m^{\frac{1}{\mathrm{l}_a m}} = a, \quad \text{and } n^{\frac{1}{\mathrm{l}_a n}} = a,$$

$$\therefore \; m^{\frac{1}{\mathrm{l}_a m}} = n^{\frac{1}{\mathrm{l}_a n}}; \qquad \text{and } m^{\mathrm{l}_a n} = n^{\mathrm{l}_a m};$$

a true result, whatever be the value of a.

$$\text{Wherefore, } m^{\log n} = n^{\log m}.$$

3. *Required from tables of logarithms calculated to a given base, as* ϵ, *to form tables of logarithms to any other base, as* 10.

$$\text{By (2),}\quad n = \epsilon^{\mathrm{l}_\epsilon n};\quad n = 10^{\mathrm{l}_{10} n};\quad 10 = \epsilon^{\mathrm{l}_\epsilon 10};$$

$$\therefore\ \epsilon^{\mathrm{l}_\epsilon n} = 10^{\mathrm{l}_{10} n} = (\epsilon^{\mathrm{l}_\epsilon 10})^{\mathrm{l}_{10} n} = \epsilon^{\mathrm{l}_\epsilon 10 \,.\, \mathrm{l}_{10} n};$$

and equating the indices of ϵ,

$$\mathrm{l}_\epsilon n = \mathrm{l}_\epsilon 10 \,.\, \mathrm{l}_{10} n;\quad \text{and}\quad \mathrm{l}_{10} n = \frac{1}{\mathrm{l}_\epsilon 10} \,.\, \mathrm{l}_\epsilon n.$$

Hence the logarithms of any number n *in two systems calculated to any bases, as* 10 *and* ϵ, *are connected by a constant multiplier,* $\left(\text{viz. } \frac{1}{\mathrm{l}_\epsilon 10}\right)$ which is called "the modulus;" and therefore from tables of logarithms calculated to a base ϵ, tables may be formed to the base 10.

Cor. By writing a for 10 in this proof, $\mathrm{l}_a n = \frac{1}{\mathrm{l}_\epsilon a} \,.\, \mathrm{l}_\epsilon n$; or, the *modulus* connecting the logarithms of a number in the two systems whose bases are a and ϵ, is $\frac{1}{\mathrm{l}_\epsilon a}$.

4. *The logarithm of the product of any number of factors is equal to the sum of the logarithms of the several factors.*

$$\text{For}\quad m \,.\, n \,.\, r \ldots = a^{\mathrm{l}_a m} \,.\, a^{\mathrm{l}_a n} \,.\, a^{\mathrm{l}_a r} \ldots = a^{(\mathrm{l}_a m + \mathrm{l}_a n + \mathrm{l}_a r + \ldots)}$$

$$\text{But}\quad m \,.\, n \,.\, r \ldots = a^{\mathrm{l}_a (mnr \ldots)};$$

$$\therefore\ \mathrm{l}_a (m \,.\, n \,.\, r \ldots) = \mathrm{l}_a m + \mathrm{l}_a n + \mathrm{l}_a r + \ldots$$

$$\text{or,}\quad \log (m \,.\, n \,.\, r \ldots) = \log m + \log n + \log r + \ldots$$

Hence, if the number be found in the tables whose logarithm is the sum of the logarithms of the several factors, the product of those factors is obtained.

5. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

$$\text{For}\quad a^{\mathrm{l}_a \left(\frac{m}{n}\right)} = \frac{m}{n} = \frac{a^{\mathrm{l}_a m}}{a^{\mathrm{l}_a n}} = a^{\mathrm{l}_a m - \mathrm{l}_a n};$$

$$\therefore\ \mathrm{l}_a\left(\frac{m}{n}\right) = \mathrm{l}_a m - \mathrm{l}_a n;\quad \text{or,}\quad \log\left(\frac{m}{n}\right) = \log m - \log n.$$

Hence, if the number be found in the tables whose logarithm is the logarithm of the dividend minus the logarithm of the divisor, the quotient is obtained.

6. *The logarithm of the* cth *power of any number is equal to* c *times the logarithm of the number;—*c *being either whole or fractional.*

For $a^{\mathrm{l}_a(m^c)} = m^c = (a^{\mathrm{l}_a m})^c = a^{c \cdot \mathrm{l}_a m}$;

$\therefore$ $\mathrm{l}_a(m^c) = c \,.\, \mathrm{l}_a m$; or, $\log(m^c) = c \,.\, \log m$.

Also, $a^{\mathrm{l}_a(m)^{\frac{1}{c}}} = m^{\frac{1}{c}} = (a^{\mathrm{l}_a m})^{\frac{1}{c}} = a^{\frac{1}{c} \cdot \mathrm{l}_a m}$;

$\therefore$ $\mathrm{l}_a(m^{\frac{1}{c}}) = \frac{1}{c} \,.\, \mathrm{l}_a m$; or, $\log(m^{\frac{1}{c}}) = \frac{1}{c} \,.\, \log m$.

Hence, the c^{th} power of any given number is that number in the tables whose logarithm is c times the logarithm of the given number; and the c^{th} root of it is that number in the tables whose logarithm is $\frac{1}{c}^{\text{th}}$ of the logarithm of the given number.

7. From the last three articles it appears that the operations of multiplication, division, involution, and evolution can be performed with greater ease by means of tables of logarithms than by the common arithmetical methods, particularly where the numbers are large. The easier arithmetical operations of addition and subtraction cannot be performed by logarithms.

8. *In the common, or Briggs' system of logarithms, where the base is* 10, *the logarithms of* $10^n \times \mathrm{N}$ *and* $\frac{\mathrm{N}}{10^n}$ *may be determined from the logarithm of* N.

$$\begin{aligned} \text{For } \mathrm{l}_{10}(10^n \times N) &= \mathrm{l}_{10} 10^n + \mathrm{l}_{10} N, \text{ by (4)} \\ &= n \,.\, \mathrm{l}_{10} 10 + \mathrm{l}_{10} N, \ldots (6) \\ &= n + \mathrm{l}_{10} N. \ldots\ldots\ldots\ldots (2.\ \text{Cor. } 1). \end{aligned}$$

$$\begin{aligned} \text{And } \mathrm{l}_{10}\left(\frac{N}{10^n}\right) &= \mathrm{l}_{10} N - \mathrm{l}_{10} 10^n, \ldots\ldots (5). \\ &= \mathrm{l}_{10} N - n. \end{aligned}$$

Thus from the pages of logarithms printed at the end of this Appendix,

$$l_{10}6 = 0{\cdot}7781513\,;$$

$$\begin{aligned}
\therefore\ l_{10}60 &= l_{10}10 + l_{10}6 &&= 1 + 0{\cdot}7781513 = 1{\cdot}7781513,\\
l_{10}600 &= l_{10}(10^2) + l_{10}6 &&= 2 + 0{\cdot}7781513 = 2{\cdot}7781513,\\
l_{10}{\cdot}6 &= l_{10}6 - l_{10}10 &&= 0{\cdot}7781513 - 1,\\
l_{10}{\cdot}006 &= l_{10}6 - l_{10}(10^3) &&= 0{\cdot}7781513 - 3.
\end{aligned}$$

The last two logarithms are thus written, $\bar{1}{\cdot}7781513$, $\bar{3}{\cdot}7781513$.

Def. The *integral* part of the logarithm is called the "*Characteristic*" of the logarithm of the number; the *decimal* part is called the "*Mantissa*"* *of the significant digits* of which the number is composed.

Thus, in $l_{10}600 = 2{\cdot}7781513$, 2 is the *Characteristic* of the *Logarithm* of the *Number* 600, ·7781513 is the *Mantissa* of the *significant digit* 6.

9. *In the common system, to determine the Characteristic of the Logarithm of any given Number.*

If a Number be between

1 and 10,	its log. is between 0 and 1;	∴ the characteristic is 0.
10 and 100,	.... 1 and 2;	∴ .... 1.
100 and 1000,	.... 2 and 3;	∴ .... 2.
10^{n-1} and 10^n	.... $n-1$ and n;	∴ .... $n-1$.

Hence the Characteristic of the Logarithm of a Number between 10^{n-1} and 10^n, (which will have n integral places) is $n - 1$, or is less by one than the number of integral places of figures which the number contains.

Again, if the number be between

1 and $\frac{1}{10}$,	its log. is between 0 and -1;	∴ the characteristic is $\bar{1}$,
$\frac{1}{10}$ and $\frac{1}{100}$,	.... -1 and -2;	∴ .... $\bar{2}$,
$\frac{1}{10^{n-1}}$ and $\frac{1}{10^n}$,	.... $-(n-1)$ and $-n$;	∴ .... $\bar{n}$.

---

* "Mantissa;" a handful thrown in over and above the exact weight; an overplus.

Hence, the Characteristic of the logarithm of a decimal fraction having $n-1$ cyphers after the decimal point, is $\bar{n}$.

Generally therefore, *the Characteristic of the Logarithm of any Number is the number of its digits minus one;* where if the number be a decimal fraction, the *cyphers* which follow the decimal point are alone counted, and are reckoned negatively.

Wherefore, conversely, if Logarithms be given having Characteristics 1, 2, 3...$\bar{1}$, $\bar{2}$, $\bar{3}$... there are in the integral parts of the numbers to which these logarithms belong 2, 3, 4......0, -1, -2... digits respectively.

Thus the Logarithms of 245 and 25400 have for Characteristics 2 and 4, and the Characteristics of the Logarithms of 2·54, 25·4, ·000254 are 0, 1, $\bar{4}$.

The Mantissa given in the tables for $l_{10}3652$ is ·5625308.

$$\therefore\ l_{10}3652 = 3\cdot5625308, \quad l_{10}36\cdot52 = 1\cdot5625308,$$

$$l_{10}365200 = 5\cdot5625308, \quad l_{10}\cdot3652 = \bar{1}\cdot5625308,$$

$$l_{10}\cdot003652 = \bar{3}\cdot5625308.$$

10. *The advantages of Briggs' system of logarithms.*

From the last Article it appears, that if the *Base* of the system be 10, it is requisite to register the *Mantissæ* only in the tables; because the *Characteristics* can be determined by counting the digits in the integral part of the numbers whose logarithms are required. This omission of the characteristics renders the common tables *less bulky* than those calculated to any other base.

Also, from (8) it appears, that in this system the Mantissa of N is also the Mantissa of $10^n \times N$, and of $\frac{N}{10^n}$,—where n is any integer: this circumstance renders the common tables *more comprehensive* than if any other base were taken; for if any other base were used, the Mantissa of N would *not* be the Mantissa of $10^n . N$, or of $\frac{N}{10^n}$.

In the same way it might be shewn that if the system of arithmetic in common use were duodecimal instead of being decimal, tables calculated to the base 12 would possess the same advantages which have been here shewn to belong to the tables in common use.

11. The tables of logarithms in common use register, some to *five*, and others to *seven*, places of decimals, the mantissæ for numbers from 1 to 100000. Two pages of logarithms are printed at the end of this Appendix, in which the mantissæ are calculated to *seven* places of decimals. The line at the top of the second of the pages begins with the number 3650 (or 36500), and its mantissa ·5622929 is placed opposite to it. And because the mantissæ of all numbers from 36500 to 36559, (comprised in the first six lines of the page), have the same initial

three figures, viz. ·562, these three figures are registered, once for all, opposite to the number 3650, and the last four figures of each succeeding mantissa are placed under the number to which they belong. Thus the first line of the page is

Num.	0	1	2	3	4	5	6	7	8	9
3650	5622929	3048	3167	3286	3405	3524	3642	3761	3880	3999

Whence is obtained, Mantissa of 36500 = ·562 2929
.... 36501 = ·562 3048
.... 36502 = ·562 3167
.... 36503 = ·562 3286,

and so on.

In the same manner, the next line gives the mantissæ of numbers from 36510 to 36519 inclusive.

12. Since a change in the value of the third figure may not take place at the *beginning* of one of the horizontal lines, whenever a mantissa is to be taken from the tables, care must be taken to get the *right* initial figures. Thus, (see p. 110), the mantissa of 36643 is ·5639910, and the last four figures of the mantissa of 36644 are put down as 0029. Now if the mantissæ of these two numbers had the *same* third figure, the mantissa of 36644 would be less than that of 36643, which cannot possibly be. A change in the value of the third figure does, in point of fact, take place here; and the mantissa of 36644, (as do those of the numbers immediately following 36644,) begins with ·564, and not with ·563.

Similar changes of the third figure of the mantissa occur at the numbers 36729, 36813, 36898, 36983, and are indicated by printing in a smaller type the fourth figure of the mantissæ of those numbers.

The construction and use of the small tables in the last column of the page will be explained hereafter.

13. EXAMPLES. (1) *To multiply 23 by 16.* Art. 4.

By the tables, p. 109., Mantissa of 23 is ·3617278,
Mantissa of 16 is ·2041200;

$$\therefore\ l_{10} 23 = 1{,}3617278$$

$$l_{10} 16 = 1\cdot 2041200$$

$$\overline{2\cdot 5658478}$$

And, p. 110, the significant digits corresponding to the mantissa ·5658478 are 3680:

$\therefore\ l_{10} 368\cdot 0$ or $l_{10} 368 = 2\cdot 5658478$; and $\therefore$ 368 is the product sought.

(2) *To multiply* ·0172 *by* ·00214.

$$\begin{array}{rl} l_{10}\,{\cdot}0172 = & \bar{2}{\cdot}2355284 \\ l_{10}\,{\cdot}00214 = & \bar{3}{\cdot}3304138, \\ \hline & \bar{5}{\cdot}5659422, \\ & = l_{10}\,{\cdot}000036808\,; \quad \text{p. 110;} \end{array}$$

Wherefore ·000036808 is the product sought.

(3) *To divide* 3672 *by* 51000. Art. 5.

$$\begin{array}{rll} l_{10}3672 = & 3{\cdot}5649027 & \text{p. 110.} \\ l_{10}51000 = & 4{\cdot}7075702 & \text{p. 109.} \\ \hline & \bar{2}{\cdot}8573325 & \\ & = l_{10}\,{\cdot}072\,; & \end{array}$$

Wherefore ·072 is the quotient sought.

(4) *To find the values of* $(15{\cdot}4)^3$, *and* $(650)^{\frac{1}{5}}$. Art. 6.

$$\begin{array}{rll} l_{10}15{\cdot}4 = & 1{\cdot}1875207 & \text{p. 109.} \\ & 3 & \\ \hline & 3{\cdot}5625621 = l_{10}3652{\cdot}3, \text{ nearly,} & \text{p. 110.} \end{array}$$

∴ 3652·3 is the approximate cube of 15·4.

$$\text{Again, } l_{10}650 = 2{\cdot}8129134, \quad \text{p. 109.}$$

$$\therefore \ \frac{1}{5}\,.\,l_{10}650 = {\cdot}5625826 = l_{10}3{\cdot}6524, \text{ nearly,} \quad \text{p. 110.}$$

∴ 3·6524 is the approximate fifth root of 650.

(5) *To find the values of* $(.085)^4$, *and of* $(.000065)^{\frac{1}{3}}$.

$$\begin{array}{l} \bar{2}{\cdot}9294189 = l_{10}\,{\cdot}085 \\ \qquad\quad 4 \\ \hline \bar{5}{\cdot}7176756 \end{array}$$

and this is the mantissa of 522006, nearly. Therefore ·0000522006 is the number sought, nearly.

$$\text{Again, } l_{10}\,{\cdot}000065 = \bar{5}{\cdot}8129134 = \bar{6} + 1{\cdot}8129134\,;$$

$$\begin{array}{rl} \therefore \ 3) & \bar{5}{\cdot}8129134 \\ \hline & \bar{2}{\cdot}6043044 = l_{10}\,{\cdot}040207, \text{ nearly.} \end{array}$$

Note. In dividing it is to be remembered that $\bar{5}{\cdot}8\ldots = \bar{6} + 1{\cdot}8\ldots$

14. *To expand* $\mathrm{l}_\epsilon(1 + \mathrm{x})$ *in a series ascending by powers of* x.

Now $a^0 = 1$, or $\mathrm{l}_\epsilon 1 = 0$; and $\mathrm{l}_\epsilon(1+x)$ becomes $\mathrm{l}_\epsilon 1$ when x vanishes; wherefore the series which expresses the value of $\mathrm{l}_\epsilon(1+x)$ cannot involve any negative power of x, (or it would become infinite when x vanished,) nor can it have a constant term, (or it would not vanish along with x).

Let then $\quad Ax + Bx^2 + Cx^3 + \ldots = \mathrm{l}_\epsilon(1+x)$ .... (1).

∴ also, by writing $x+h$ instead of x,

$$A\,(x+h + B\,(x+h)^2 + C\,(x+h)^3 + \ldots = \mathrm{l}_\epsilon(1+\overline{x+h}).$$

By subtraction,

$$A\,.\,\{(x+h)-x\} + B\,.\,\{(x+h)^2 - x^2\} + C\,.\,\{(x+h)^3 - x^3\} + \ldots$$

or $$Ah + B\,.\,(2xh + h^2) + C\,.\,(3x^2h + 3xh^2 + h^3) + \ldots$$

$$= \mathrm{l}_\epsilon(1 + \overline{x+h}) - \mathrm{l}_\epsilon(1+x)$$

$$= \mathrm{l}_\epsilon\frac{1+\overline{x+h}}{1+x}, \text{ by Art. 5}, \quad = \mathrm{l}_\epsilon\left(1 + \frac{h}{1+x}\right)$$

$$= A\,.\,\frac{h}{1+x} + B\,.\left(\frac{h}{1+x}\right)^2 + C\,.\left(\frac{h}{1+x}\right)^3 + \ldots \text{ by (1).}$$

By dividing both sides of this equation by h,

$$A + B\,.\,(2x+h) + C\,.\,(3x^2 + 3xh + h^2) + \ldots = \frac{A}{1+x} + B\,.\,\frac{h}{(1+x)^2} + \ldots$$

And making $h = 0$,

$$A + 2Bx + 3Cx^2 + \ldots = \frac{A}{1+x} = A\,.\,(1 - x + x^2 - x^3 + \ldots)$$

Equating the coefficients of like powers of x,

$$2B = -A, \qquad 3C = A, \qquad 4D = -A, \qquad \&c.$$

$$\therefore\ B = -\tfrac{1}{2}A, \qquad C = \tfrac{1}{3}A, \qquad D = -\tfrac{1}{4}A, \qquad \&c.$$

$$\therefore\ \mathrm{l}_\epsilon(1+x) = A\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) \ldots\ldots\ldots\ldots\ldots\ldots \text{(i).}$$

Cor. 1. Since A is a constant quantity which is entirely dependent on the value of a, let ϵ be the quantity whose value is such that $\epsilon^{\frac{1}{A}} = a$;

$$\therefore \frac{1}{A} = \mathrm{l}_\epsilon a; \quad \text{and } \mathrm{l}_a(1+x) = \frac{1}{\mathrm{l}_\epsilon a} \cdot \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right) \ldots\ldots\ldots\ldots \text{(ii.)}$$

and if the base be ϵ,

$$\mathrm{l}_\epsilon(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots; \qquad \ldots\ldots\ldots \text{(iii.)}$$

The value of ϵ, (which is a constant quantity, 2·7182818...) will be found hereafter.

Logarithms to the base ϵ are called *Napierian*, from *Napier*, the inventor of logarithms, who adopted this base because logarithms to it are more easily calculated than those to any other base; as is evident from comparing the series (ii.) and (iii.)

Cor. 2. $\mathrm{l}_\epsilon a = \mathrm{l}_\epsilon\{1 + (a-1)\}$, which by (iii.) is

$$= (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \ldots\ldots\ldots\ldots \text{(iv.)}$$

Cor. 3. $$A = \frac{1}{\mathrm{l}_\epsilon a} = \frac{1}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \ldots}.$$

Cor. 4. The modulus $\dfrac{1}{\mathrm{l}_\epsilon 10}$ of the common system of logarithms may be calculated thus; by taking for granted the formulæ (ix.) and (vi.) which are proved hereafter.

$$\text{By (ix.), } \mathrm{l}_\epsilon 5 = \mathrm{l}_\epsilon 4 + 2 \cdot \left\{\frac{1}{8+1} + \frac{1}{3} \cdot \frac{1}{(8+1)^3} + \frac{1}{5} \cdot \frac{1}{(8+1)^5} + \ldots\right\}$$

$$= 2 \cdot \mathrm{l}_\epsilon 2 + 2 \cdot \left\{\frac{1}{9} + \frac{1}{3} \cdot \frac{1}{9^3} + \frac{1}{5} \cdot \frac{1}{9^5} + \ldots\right\};$$

$$\therefore \mathrm{l}_\epsilon 10, = \mathrm{l}_\epsilon 2 + \mathrm{l}_\epsilon 5, = 3 \cdot \mathrm{l}_\epsilon 2 + 2 \cdot \left\{\frac{1}{9} + \frac{1}{3} \cdot \frac{1}{9^3} + \frac{1}{5} \cdot \frac{1}{9^5} + \ldots\right\}$$

$$= \begin{cases} 6 \cdot \left\{\frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3^3} + \frac{1}{5} \cdot \frac{1}{3^5} + \ldots\right\} \text{ by (vi.)} \\ + 2 \cdot \left\{\frac{1}{9} + \frac{1}{3} \cdot \frac{1}{9^3} + \frac{1}{5} \cdot \frac{1}{9^5} + \ldots\right\} \end{cases}$$

$$= 2 \cdot \left\{\begin{matrix} 1 + \frac{1}{3} \cdot \frac{1}{9} + \frac{1}{5} \cdot \frac{1}{9^2} + \ldots \\ + \frac{1}{9} + \frac{1}{3} \cdot \frac{1}{9^3} + \frac{1}{5} \cdot \frac{1}{9^5} + \ldots \end{matrix}\right\}$$

$$= 2 \,.\, 30258509\ldots$$

Whence is obtained, $\dfrac{1}{\mathrm{l}_\epsilon 10} = 0·434294819\ldots\ldots$

15. Some rapidly-converging formulæ for the calculation of logarithms will next be investigated.

I. By (iii.), $l_e \frac{1}{x} = l_e\left(1 - \frac{x-1}{x}\right) = -\frac{x-1}{x} - \frac{1}{2}\left(\frac{x-1}{x}\right)^2 - \dots$

But $l_e \frac{1}{x} = l_e 1 - l_e x = -l_e x$;

$$\therefore\ l_e x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \dots\dots\dots \text{(v.)}$$

II. Again; $l_e(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$; and $l_e(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \dots$;

$\therefore\ l_e(1+z) - l_e(1-z)$, or $l_e\frac{1+z}{1-z}$, $= 2\,.\left\{z + \frac{z^3}{3} + \frac{z^5}{5} + \dots\right\}$.

And if $\frac{1+z}{1-z} = x$, and $\therefore\ z = \frac{x-1}{x+1}$, this becomes,

$$l_e x = 2\,.\left\{\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \dots\right\} \dots\dots\dots \text{(vi.)}$$

If x be a little greater than 1, this series converges very rapidly.

III. Again, $l_e x = l_e (x^{\frac{1}{m}})^m = m\,.\,l_e x^{\frac{1}{m}}$

$= m\,.\left\{(x^{\frac{1}{m}} - 1) - \frac{1}{2}\,.\,(x^{\frac{1}{m}} - 1)^2 + \dots\right\}$, by (iv.)

and, (x being greater than 1,) by assuming m of sufficient magnitude $x^{\frac{1}{m}}$ may be made to differ from 1 by any definite quantity, however small the quantity may be; in which case the succeeding terms of the series may be neglected as being of inconsiderable magnitude with respect to the first, and

$$l_e x = m\,.\,(x^{\frac{1}{m}} - 1) \dots\dots\dots \text{(vii.)}$$

16. *Having given* $l_e x$, *to find* $l_e(x+z)$; *z being small when compared with* x.

$$l_e(x+z) = l_e\left\{x\,.\left(1 + \frac{z}{x}\right)\right\} = l_e x + l_e\left(1 + \frac{z}{x}\right).$$

Expanding $l_e\left(1 + \frac{z}{x}\right)$ by (vi.), $\left[\text{since } \dfrac{\left(1 + \frac{z}{x}\right) - 1}{\left(1 + \frac{z}{x}\right) + 1} = \dfrac{z}{2x + z}\right]$,

$$l_e(x+z) = l_e x + 2\left\{\frac{z}{2x+z} + \frac{1}{3}\,.\left(\frac{z}{2x+z}\right)^3 + \dots\right\} \dots\dots\dots \text{(viii.)}$$

Cor. If $z = 1$; $l_e(1+x) = l_e x + 2\left\{\frac{1}{2x+1} + \frac{1}{3}\cdot\frac{1}{(2x+1)^3} + \frac{1}{5}\cdot\frac{1}{(2x+1)^5} + \dots\right\} \dots \text{(ix.)}$

which is useful in computing $l_e(1+x)$ from $l_e x$, particularly when x is large.

17. *Having given the Napierian Logarithms of two successive numbers,* x − 1 *and* x, *to find that of the number next following.*

$$l_e(x+1) = l_e\frac{x^2-1}{x-1} = l_e\frac{x^2.\left(1-\frac{1}{x^2}\right)}{x-1}$$

$$= 2l_e x - l_e(x-1) + l_e\left(1-\frac{1}{x^2}\right).$$

Expanding $\left(1-\frac{1}{x^2}\right)$ by (vi.), [since $\dfrac{\left(1-\frac{1}{x^2}\right)-1}{\left(1-\frac{1}{x^2}\right)+1} = -\dfrac{1}{2x^2-1}$],

$$l_e(x+1) = 2.l_e x - l_e(x-1) - 2.\left\{\frac{1}{2x^2-1} + \frac{1}{3}.\frac{1}{(2x^2-1)^3} + \dots\right\}\text{..................(x.)}$$

18. *To expand* a^x *in a series ascending by powers of* x; i. e. *to expand the Number in a series ascending by powers of the Logarithm.*

$$a^x = \{1+(a-1)\}^x$$

$$= 1 + x.(a-1) + x.\frac{x-1}{2}.(a-1)^2 + x.\frac{x-1}{2}.\frac{x-2}{3}.(a-1)^3 + \dots.$$

let the coefficient of x {which is $(a-1) - \frac{1}{2}.(a-1)^2 + \frac{1}{3}.(a-1)^3 - \dots$} be represented by p_1, and let the coefficients of x^2, x^3, ... be represented by p_2, p_3, ... ; then

$$a^x = 1 + p_1x + p_2x^2 + p_3x^3 + \dots$$

$$\therefore a^z = 1 + p_1z + p_2z^2 + p_3z^3 + \dots$$

Now $a^{(x+z)} = a^x.a^z$,

$$\text{or}\quad 1 + p_1.(x+z) + p_2.(x+z)^2 \dots + p_n.(x+z)^n + \dots$$

$$= \{1 + p_1x + p_2x^2 + \dots + p_nx^n + \dots\} \times \{1 + p_1z + p_2z^2 + \dots + p_nz^n + \dots\},$$

and by equating the coefficients of the terms involving xz, x^2z, x^3z, ... $x^{n-1}z$, ...

$$2p_2 = p_1.p_1, \qquad \therefore p_2 = \frac{p_1^2}{2}$$

$$3p_3 = p_1.p_2, \qquad \therefore p_3 = \frac{p_1^3}{1.2.3}$$

$$4\,p_4 = p_1 \,.\, p_3, \qquad \therefore\ p_4 = \frac{p_1^{\,4}}{1\,.\,2\,.\,3\,.\,4}$$

$$\dots\dots\dots\dots\dots \qquad \dots\dots\dots\dots\dots$$

$$n\,.\,p_n = p_1\,.\,p_{n-1}, \qquad \therefore\ p_n = \frac{p_1^{\,n}}{1\,.\,2\,.\,3\dots n}\,.$$

Now $p_1 = (a-1) - \tfrac{1}{2}\,.\,(a-1)^2 + \tfrac{1}{3}\,.\,(a-1)^3 - \dots = \mathrm{l}_\epsilon a;$ by (iv.)

$$\therefore\ a^x = 1 + \frac{\mathrm{l}_\epsilon a}{1}\,.\,x + \frac{(\mathrm{l}_\epsilon a)^2}{1\,.\,2}\,.\,x^2 + \frac{(\mathrm{l}_\epsilon a)^3}{1.2.3}\,.\,x^3 + \dots + \frac{(\mathrm{l}_\epsilon a)^n}{1.2.3\dots n}\,.\,x^n + \dots \quad \text{(xi.)}$$

Cor. If the base be ϵ, (xi) becomes

$$\epsilon^x = 1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \dots + \frac{x^n}{1\,.\,2\dots n} + \dots\dots\dots\dots \quad \text{(xii.)}$$

19. *To find the value of ϵ, the base of the Napierian system of logarithms.*

If $x = 1$, and the base be ϵ, the series for a^x in the last Article becomes

$$\epsilon = 1 + 1 + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \dots + \frac{1}{1\,.\,2\,.\,3\dots n} + \dots$$

Now,
$$\begin{aligned}
1 + 1 + \frac{1}{1\,.\,2} &= 2{\cdot}5 \\
\frac{1}{1\,.\,2\,.\,3} &= {\cdot}1666666666 \\
\frac{1}{1\,.\,2\,.\,3\,.\,4} &= {\cdot}0416666666 \\
\frac{1}{1\,.\,2\,.\,3\,.\,4\,.\,5} &= {\cdot}0083333333 \\
\frac{1}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6} &= {\cdot}0013888888 \\
\frac{1}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7} &= {\cdot}0001984126 \\
\frac{1}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8} &= .0000248015 \\
\frac{1}{1\,.\,2\,.\,3\,\dots\dots\dots\dots\,8\,.\,9} &= {\cdot}0000027557 \\
\frac{1}{1\,.\,2\,.\,3\,.\,4\dots\dots\dots 9\,.\,10} &= {\cdot}0000002755 \\
\hline
&\quad 2{\cdot}7182818
\end{aligned}$$

This result gives the correct value of ϵ, the base of the Napierian system, so far as the figures are put down. By taking more terms and a greater number of figures in each, the value of ϵ might be determined to any degree of accuracy required.

20. *On the construction of the common tables of logarithms.*

From one of the series (v), (vi), (vii), the Napierian logarithms of low prime numbers may be found. The logarithm of a high number which is not a prime may be determined from the logarithms of its factors, by resolving the number into powers of its prime factors;

Thus, $\mathrm{l}_\epsilon 288 = \mathrm{l}_\epsilon 2^5 . 3^2 = \mathrm{l}_\epsilon 2^5 + \mathrm{l}_\epsilon 3^2 = 5 \times \mathrm{l}_\epsilon 2 + 2 \times \mathrm{l}_\epsilon 3$.

And the expressions (viii), (ix), (x), will greatly facilitate the finding of the logarithms of high numbers which are not primes.

The Napierian logarithms having been determined, the tables to base 10 are deduced from them by multiplying each by $\frac{1}{\mathrm{l}_\epsilon 10}$, which is equal to ·4342944819... Arts. 3 and 14. Cor. 4.

Some of the artifices used in computing the tables may be found in Sharpe's "Method of making Logarithms" prefixed to Sherwin's Tables.

21. In the common tables are registered the mantissæ of the logarithms of numbers of five places of figures, these mantissæ being computed to seven places of decimals. At the side of each page is placed a "Table of proportional parts," by which, as it will be shewn, the mantissa may be found of the logarithm of a number containing six or seven places of figures; and conversely, if a logarithm be given whose mantissa is not contained exactly in the tables, the number corresponding to it may be determined, by means of these additional tables, to six or seven places of figures.

22. *On the construction and use of the Tables of Proportional Parts.*

Let m_1 and m_2 be the mantissæ of two *consecutive* integral numbers, n and $n+1$, which contain five digits each; and let m be the mantissa of the number $n + \frac{a}{10}$, which contain *six* digits, the last of which (a) is after the decimal point.

Now, since n and $n + \frac{a}{10}$ have the same number of integral places, their logarithms have the same characteristic;

$$\therefore \quad m - m_1 = \mathrm{l}_{10}\left(n + \frac{a}{10}\right) - \mathrm{l}_{10} n = \mathrm{l}_{10} \frac{n + \frac{a}{10}}{n}, \text{ Art. 5. } = \mathrm{l}_{10}\left(1 + \frac{a}{10n}\right)$$

which, by expanding $\mathrm{l}_{10}\left(1 + \frac{a}{10n}\right)$ by (ii.) and neglecting the succeeding terms as being small compared with the first term, becomes $\frac{1}{\mathrm{l}_\epsilon 10} \cdot \frac{a}{10n}$, nearly.

$$\text{Similarly,}\quad m_2 - m_1 = l_{10}(n+1) - l_{10}n = \frac{1}{l_e 10}\cdot\frac{1}{n};$$

$$\therefore\quad m - m_1 = (m_2 - m_1)\cdot\frac{a}{10}.$$

Whence m may be found, if m_1, m_2, a, be given; or a may be determined, when m_1, m_2, m, are known.

23. By the Tables, page 110,

$$\begin{aligned} \text{Mantissa of } 36633 &= m_2 = \cdot 5638725 \\ \ldots\ldots\ldots\ldots\ldots\ 36632 &= m_1 = \cdot 5638606 \\ \hline \therefore\ m_2 - m_1 &= \cdot 0000119 \end{aligned}$$

And in the expression $m - m_1 = (m_2 - m_1)\frac{a}{10}$, writing for a the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 successively, the former of these tables is obtained.

a	$m_2 - m_1$	
1	$\cdot 0000119 \times \frac{1}{10}$	
	or ·00000119	·0000012 nearly
2	·00000238	·0000024 ....
3	·00000357	·0000036 ....
4	·00000476	·0000048 ....
5	·00000595	·0000060 ....
6	·00000714	·0000071 ....
7	·00000833	·0000083 ....
8	·00000952	·0000095 ... ...
9	·00001071	·0000107 ....

119	
1	12
2	24
3	36
4	48
5	60
6	71
7	83
8	95
9	107

Now, the "difference" put down in the Tables for numbers near 36600 is 119, and the "Table of Proportional Parts" is the latter of the Tables above.

It appears then, that *the significant digits only* of the whole difference, and of the differences corresponding to the several digits, are inserted in the table of proportional parts. Hence the following Rule for constructing Tables of Proportional Parts is evident:

Of the *significant* part of the whole difference point off the last digit as a decimal; (this is the same thing as multiplying the whole difference by $\frac{1}{10}$, the whole difference being treated as an integer). Multiply the resulting number by 1, 2, 3, ... 9 successively, and the *whole* numbers thus obtained (the last digit in the integral part being increased by unity where the decimal part is not less than ·5) are the significant parts of the differences for the digits respectively.

Thus, let the significant part of the whole difference be 156.

15·6 × 1 = 15·6 = 16 nearly.
... × 2 = 31·2 = 31
... × 3 = 46·8 = 47
... × 4 = 62·4 = 62
... × 5 = 78·0 = 78

15·6 × 6 = 93·6 = 94 nearly.
... × 7 = 109·2 = 109
... × 8 = 124·8 = 125
... × 9 = 140·4 = 140.

If the digit be given the difference is immediately known from such a table, or vice versâ. To avoid the necessity of performing the operation of subtraction in any particular case in order to find the whole difference, there is a line in the tables marked at the top with "Diff.", in which the difference is placed opposite to that logarithm at which such difference begins. To know what the difference therefore is in any particular case, it is merely requisite to take the number in this line next above the logarithm in question.

Ex. 1. *To find the Number whose Logarithm is 3·5677766.*

By the tables, p. 110, the mantissa next below the given mantissa is that of $l_{10}36963$, and the whole difference put down is 117.

$$
\begin{array}{lll}
\text{Mantissa of the given Logarithm} & = m & = \cdot 5677766 \\
\text{Mantissa of } l_{10}36963 & = m_1 & = \cdot 5677672 \\
\hline
& \therefore\ m - m_1 = & 94
\end{array}
$$

By the table of Proportional Parts to "Diff." 117, the difference 94 corresponds to the digit 8; therefore the significant part of the Number sought is 369638. Also, since the given Logarithm has 3 for its characteristic, the Number required is 3696·38.

Ex. 2. *Required the Logarithm of 367·654.*

$$
\begin{array}{lrl}
\text{By p. 110,} & l_{10}367\cdot 650 & = 2\cdot 5654346 \\
\text{And "Diff." being 118, Part for} & 4 & = 47 \\
\hline
& \therefore\ l_{10}367\cdot 654 & = 2.5654393
\end{array}
$$

24. *To find the Mantissa of the Logarithm of a Number which has seven places of digits.*

Let m_1 and m_2 be the mantissæ of n and $n+1$, two successive integers of five digits each: let M be that of $n+\frac{a}{10}+\frac{b}{100}$, which has the same number of integral places as n and $n+1$, and has also one digit (a) in the place of the tenths, and another (b) in that of the hundredths.

Since a and b are digits after the decimal point, the numbers n and $n+\frac{a}{10}+\frac{b}{100}$ have the same number of integral places, their logarithms have the same characteristic, and the difference therefore of their mantissæ is the same as the difference of their logarithms;

$$\therefore\quad M-m_1=\mathrm{l}_{10}\left(n+\frac{a}{10}+\frac{b}{100}\right)-\mathrm{l}_{10}\,n=\mathrm{l}_{10}\frac{n+\frac{a}{10}+\frac{b}{100}}{n}$$

$$=\mathrm{l}_{10}\left\{1+\frac{1}{n}\cdot\left(\frac{a}{10}+\frac{b}{100}\right)\right\}=\frac{1}{\mathrm{l}_{\epsilon}10}\cdot\frac{1}{n}\cdot\left(\frac{a}{10}+\frac{b}{100}\right)\text{ by (ii),}$$

neglecting the terms of the series after the first.

Similarly, since n and $n+1$ are integers of the same number of digits,

$$m_2-m_1=\mathrm{l}_{10}(n+1)-\mathrm{l}_{10}\,n=\mathrm{l}_{10}\left(1+\frac{1}{n}\right)=\frac{1}{\mathrm{l}_{\epsilon}10}\cdot\frac{1}{n};$$

$$\therefore\quad M-m_1=(m_2-m_1)\cdot\left(\frac{a}{10}+\frac{b}{100}\right)=(m_2-m_1)\cdot\frac{a}{10}+\frac{1}{10}\cdot\left\{(m_2-m_1)\cdot\frac{b}{10}\right\}.$$

Now the first part of the expression, $(m_2-m_1)\cdot\frac{a}{10}$, is the quantity to be added to m_1 for the *first* additional digit a, Art 22. And $(m_2-m_1)\cdot\frac{b}{10}$ is what *would* have been added, had b been the *first* additional digit, instead of being the *second;* wherefore the quantity which is to be added for b when it is the *second* additional digit, namely, $\frac{1}{10}\cdot\left\{(m_2-m_1)\cdot\frac{b}{10}\right\}$, is the tenth part of what would have been added, had b been the *first*, instead of being the *second*, additional figure.

The following Examples will explain what has been said.

Ex. 1. *To find* $\mathrm{l}_{10}\,3684{\cdot}286$.

By the tables, p. 110, $\mathrm{l}_{10}3684{\cdot}2=3{\cdot}5663432$, and "Diff." being 118, the part for a *first* additional digit 8 is 94; and for a *first* additional digit 6 the part is 71, and therefore the part when 6 is a *second* additional digit is $\frac{71}{10}$, or 7·1. The operation is thus performed;

$$\begin{array}{lrr}
 & \mathrm{l}_{10}\,3684{\cdot}2 & =\ 3{\cdot}5663432\\
\text{Part for} & 8\ = & 94\\
\text{Part for} & 6\ = & 7{\cdot}1\\
\hline
\therefore & \mathrm{l}_{10}\,3684{\cdot}286 & =\ 3{\cdot}5663533
\end{array}$$

Ex. 2. *To find the Number whose Logarithm is 2·5656560.*

$$\begin{array}{rrl} & 2{\cdot}5656560 & \\ \text{Now} & \underline{2{\cdot}5656471} & = \mathrm{l}_{10}\,367{\cdot}83, \quad \text{p. } 110. \\ & 89 & \\ & \underline{83} & = \text{Part for digit } 7. \quad \text{“Diff.” being } 118. \\ & \underline{\underline{6}} & = \text{Part for a second digit } 5. \\ \therefore & 2{\cdot}5656560 & = \mathrm{l}_{10}\,367{\cdot}8375, \end{array}$$

and therefore 367·8375 is the number sought.

25. *On the adaptation of formulæ to logarithmic computation.*

After a table of logarithms has once been constructed, the labour of certain arithmetical operations can be materially diminished, while at the same time the chance of committing errors is lessened. But by referring to Articles 4, 5, 6 of this Appendix, it will appear that the sole arithmetical operations which can be performed by logarithms, are those of Multiplication, Division, Involution, and Evolution. Before, therefore, the value of an expression can be calculated by means of logarithms, the expression must be put into such a form that no other arithmetic operations than these have to be performed. Such an arrangement of an expression is called the *adaptation* of it *to logarithmic computation.*

Thus if a, b, c, the three sides of a triangle, be given, logarithms cannot be *directly* applied to determine the value of $\cos A$ from the equation,

$$\operatorname{Cos} A = \frac{b^2+c^2-a^2}{2bc};$$

but if from this equation the formula,

$$\operatorname{Cos} \tfrac{1}{2} A = \sqrt{\frac{S.(S-a)}{bc}}, \text{ where } S = \tfrac{1}{2}(a+b+c),$$

be deduced, logarithms can be immediately applied to determine the value of $\operatorname{Cos} \frac{1}{2} A$.

For a, b, c being given, S and $S-a$ are easily determined; and these being known, $\operatorname{Cos} \frac{1}{2} A$ is determined from the equation

$$\mathrm{l}_{10} \operatorname{Cos} \tfrac{1}{2} A = \tfrac{1}{2}\{\mathrm{l}_{10}[S.(S-a)] - \mathrm{l}_{10} bc\} = \tfrac{1}{2}\{\mathrm{l}_{10} S + \mathrm{l}_{10}(S-a) - \mathrm{l}_{10} b - \mathrm{l}_{10} c\}.$$

The two accompanying pages of logarithms are taken from Babbage's Tables, the most correct and the best arranged of any which have been published. The columns have been omitted which in those Tables are given to determine the number of seconds in an angle containing a given number of degrees, minutes, and seconds, and conversely.

1	0000000	51	7075702	101	0043214	151	1789769	201	3031961
2	3010300	52	7160033	102	0086002	152	1818436	202	3053514
3	4771213	53	7242759	103	0128372	153	1846914	203	3074960
4	6020600	54	7323938	104	0170333	154	1875207	204	3096302
5	6989700	55	7403627	105	0211893	155	1903317	205	3117539
6	7781513	56	7481880	106	0253059	156	1931246	206	3138672
7	8450980	57	7558749	107	0293838	157	1958997	207	3159703
8	9030900	58	7634280	108	0334238	158	1986571	208	3180633
9	9542425	59	7708520	109	0374265	159	2013971	209	3201463
10	0000000	60	7781513	110	0413927	160	2041200	210	3222193
11	0413927	61	7853298	111	0453230	161	2068259	211	3242825
12	0791812	62	7923917	112	0492180	162	2095150	212	3263359
13	1139434	63	7993405	113	0530784	163	2121876	213	3283796
14	1461280	64	8061800	114	0569049	164	2148438	214	3304138
15	1760913	65	8129134	115	0606978	165	2174839	215	3324385
16	2041200	66	8195439	116	0644580	166	2201081	216	3344538
17	2304489	67	8260748	117	0681859	167	2227165	217	3364597
18	2552725	68	8325089	118	0718820	168	2253093	218	3384565
19	2787536	69	8388491	119	0755470	169	2278867	219	3404441
20	3010300	70	8450980	120	0791812	170	2304489	220	3424227
21	3222193	71	8512583	121	0827854	171	2329961	221	3443923
22	3424227	72	8573325	122	0863598	172	2355284	222	3463530
23	3617278	73	8633229	123	0899051	173	2380461	223	3483049
24	3802112	74	8692317	124	0934217	174	2405492	224	3502480
25	3979400	75	8750613	125	0969100	175	2430380	225	3521825
26	4149733	76	8808136	126	1003705	176	2455127	226	3541084
27	4313638	77	8864907	127	1038037	177	2479733	227	3560259
28	4471580	78	8920946	128	1072100	178	2504200	228	3579348
29	4623980	79	8976271	129	1105897	179	2528530	229	3598355
30	4771213	80	9030900	130	1139434	180	2552725	230	3617278
31	4913617	81	9084850	131	1172713	181	2576786	231	3636120
32	5051500	82	9138139	132	1205739	182	2600714	232	3654880
33	5185139	83	9190781	133	1238516	183	2624511	233	3673559
34	5314789	84	9242793	134	1271048	184	2648178	234	3692159
35	5440680	85	9294189	135	1303338	185	2671717	235	3710679
36	5563025	86	9344985	136	1335389	186	2695129	236	3729120
37	5682017	87	9395193	137	1367206	187	2718416	237	3747483
38	5797836	88	9444827	138	1398791	188	2741578	238	3765770
39	5910646	89	9493900	139	1430148	189	2764618	239	3783979
40	6020600	90	9542425	140	1461280	190	2787536	240	3802112
41	6127839	91	9590414	141	1492191	191	2810334	241	3820170
42	6232493	92	9637878	142	1522883	192	2833012	242	3838154
43	6334685	93	9684829	143	1553360	193	2855573	243	3856063
44	6434527	94	9731279	144	1583625	194	2878017	244	3873898
45	6532125	95	9777236	145	1613680	195	2900346	245	3891661
46	6627578	96	9822712	146	1643529	196	2922561	246	3909351
47	6720979	97	9867717	147	1673173	197	2944662	247	3926970
48	6812412	98	9912261	148	1702617	198	2966652	248	3944517
49	6901961	99	9956352	149	1731863	199	2988531	249	3961993
50	6989700	100	0000000	150	1760913	200	3010300	250	3979400

Num.	0	1	2	3	4	5	6	7	8	9	Diff.	
3650	5622929	3048	3167	3286	3405	3524	3642	3761	3880	3999	119	119 1 12 2 24 3 36 4 48 5 60 6 71 7 83 8 95 9 107
1	4118	4237	4356	4475	4594	4713	4832	4951	5070	5189		
2	5308	5427	5546	5664	5783	5902	6021	6140	6259	6378		
3	6497	6616	6734	6853	6972	7091	7210	7329	7448	7567		
4	7685	7804	7923	8042	8161	8280	8398	8517	8636	8755		
5	8874	8993	9111	9230	9349	9468	9587	9705	9824	9943		
6	5630062	0181	0299	0418	0537	0656	0775	0893	1012	1131		
7	1250	1368	1487	1606	1725	1843	1962	2081	2200	2318		
8	2437	2556	2674	2793	2912	3031	3149	3268	3387	3505		
9	3624	3743	3861	3980	4099	4218	4336	4455	4574	4692		
3660	4811	4930	5048	5167	5285	5404	5523	5641	5760	5879		
1	5997	6116	6235	6353	6472	6590	6709	6828	6946	7065		
2	7183	7302	7421	7539	7658	7776	7895	8013	8132	8251		
3	8369	8488	8606	8725	8843	8962	9081	9199	9318	9436		
4	9555	9673	9792	9910	0029	0147	0266	0384	0503	0621		
5	5640740	0858	0977	1095	1214	1332	1451	1569	1688	1806	118	118 1 12 2 24 3 35 4 47 5 59 6 71 7 83 8 94 9 106
6	1925	2043	2162	2280	2398	2517	2635	2754	2872	2991		
7	3109	3228	3346	3464	3583	3701	3820	3938	4056	4175		
8	4293	4412	4530	4648	4767	4885	5004	5122	5240	5359		
9	5477	5595	5714	5832	5951	6069	6187	6306	6424	6542		
3670	6661	6779	6897	7016	7134	7252	7371	7489	7607	7726		
1	7844	7962	8080	8199	8317	8435	8554	8672	8790	8908		
2	9027	9145	9263	9382	9500	9618	9736	9855	9973	0091		
3	5650209	0328	0446	0564	0682	0800	0919	1037	1155	1273		
4	1392	1510	1628	1746	1864	1983	2101	2219	2337	2455		
5	2573	2692	2810	2928	3046	3164	3282	3401	3519	3637		
6	3755	3873	3991	4109	4228	4346	4464	4582	4700	4818		
7	4936	5054	5173	5291	5409	5527	5645	5763	5881	5999		
8	6117	6235	6353	6471	6590	6708	6826	6944	7062	7180		
9	7298	7416	7534	7652	7770	7888	8006	8124	8242	8360		
3680	8478	8596	8714	8832	8950	9068	9186	9304	9422	9540		
1	9658	9776	9894	0012	0130	0248	0366	0484	0602	0720		
2	5660838	0956	1074	1192	1310	1428	1545	1663	1781	1899		
3	2017	2135	2253	2371	2489	2607	2725	2843	2960	3078		
4	3196	3314	3432	3550	3668	3786	3903	4021	4139	4257		
5	4375	4493	4611	4728	4846	4964	5082	5200	5318	5435		
6	5553	5671	5789	5907	6025	6142	6260	6378	6496	6614		
7	6731	6849	6967	7085	7203	7320	7438	7556	7674	7791		
8	7909	8027	8145	8262	8380	8498	8616	8733	8851	8969		
9	9087	9204	9322	9440	9557	9675	9793	9911	0028	0146		
3690	5670264	0381	0499	0617	0734	0852	0970	1087	1205	1323		
1	1440	1558	1676	1793	1911	2029	2146	2264	2382	2499		
2	2617	2735	2852	2970	3087	3205	3323	3440	3558	3675		
3	3793	3911	4028	4146	4263	4381	4499	4616	4734	4851		
4	4969	5086	5204	5322	5439	5557	5674	5792	5909	6027	117	117 1 12 2 23 3 35 4 47 5 59 6 70 7 82 8 94 9 105
5	6144	6262	6379	6497	6615	6732	6850	6967	7085	7202		
6	7320	7437	7555	7672	7790	7907	8025	8142	8260	8377		
7	8495	8612	8729	8847	8964	9082	9199	9317	9434	9552		
8	9669	9787	9904	0021	0139	0256	0374	0491	0608	0726		
9	5680843	0961	1078	1196	1313	1430	1548	1665	1782	1900		
	0	1	2	3	4	5	6	7	8	9		

APPENDIX II.

ON THE CONSTRUCTION AND USE OF TABLES OF GONIOMETRIC RATIOS.

1. If it be required to find the value of a trigonometrical formula in which the sines, cosines, tangents, secants, &c. of given angles enter, much labour may evidently be avoided by determining the values of these quantities once for all and registering them in tables.

In very small angles the sines and tangents are exceedingly small quantities, and if they be expressed as decimal fractions, two or three cyphers will follow the decimal point before the significant digits are arrived at. Now in order to avoid the inconvenience of printing these cyphers, the real values of all Goniometrical Ratios are multiplied by 10,000 (or the decimal point is moved four places to the right) before they are registered in the tables; and the tables so formed are *called*, "Tables of natural sines, cosines, &c.*"

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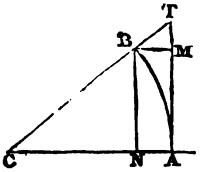
* The tables of Goniometric Ratios are sometimes said to be "calculated to a radius of 10,000." To explain the meaning of this expression.

If with C as centre, and radius CA, an arc AB be described, and BN, AT be ⊥ to CA, the *lines* NB, CN, AT, CT, AN respectively, are sometimes defined to be the *sine*, *cosine*, *tangent*, *secant*, and *versed sine*, *of the angle* ACB *to the radius* CA,—or the *sine*, *cosine*, *tangent*, *secant*, and *versed sine*, *of the arc* AB.



The expression "to the radius CA" is necessary to these definitions, because the lines NB, AT... depend on the magnitude of CA as well as upon that of the angle ACB; in fact, if the angle ACB continue of the same magnitude, those lines vary directly as CA.

Now

2. *To find the sine and cosine of* 10″.

Let θ be the Circular Measure of an angle of 10″.

$$\text{Then}\quad \frac{\theta}{\pi} = \frac{10}{180 \times 60 \times 60} = \frac{1}{64800};$$

$$\therefore\quad \theta = \cdot000048481368110.$$

$$\text{But}\quad \text{Sin}\,\theta > \theta - \tfrac{1}{4}\,\theta^3, \qquad (\text{Art. } 105).$$

∴ , a fortiori,

$$\begin{aligned}\text{Sin } 10'' &> \theta - \tfrac{1}{4}\,(\cdot00005)^3 \\ &> \cdot000048481368110 \\ &\quad - \cdot000000000000032 \\ &> \cdot000048431368078.\end{aligned}$$

$$\text{Also,}\quad \text{Sin } 10'' < \theta, \quad \text{or}\quad \cdot000048481368110.$$

Wherefore a near approximation to the value of Sin 10″ is obtained by taking the first twelve places which these two quantities have in common, and therefore

$$\text{Sin } 10'' = \cdot000048181368 \text{ very nearly.}$$

By substituting this value of Sin 10″ in the formula

$$\text{Cos } 10'' = \sqrt{\{1 - \sin^2 10''\}},$$

there is obtained

$$\text{Cos } 10'' = \cdot999999988248.$$

Now by the definition which has been adopted of the sine,

$$\text{Sin } ACB = \frac{NB}{CB} = NB \cdot \frac{10{,}000}{CA}.$$

$$\therefore\quad 10{,}000 \times \sin ACB = NB \cdot \frac{10{,}000}{CA}.$$

But 10,000 × sin ACB is the tabular sine of ABC;

$$\therefore\quad \text{tab. } \sin ACB = NB \cdot \frac{10{,}000}{CA} = NB, \quad \text{if } CA = 10{,}000.$$

Wherefore the tab. sine of ACB expresses the magnitude of the line NB, (the Sine of ACB to the radius CA), the magnitude of CA being represented by 10,000.

Similarly the tab. cosine, tab. tangent, &c., of ACB, express the magnitudes of the lines CN, AT, &c., the magnitude of CA being 10,000.

3. *The Sine and Cosine of* 10″ *being known, the Sines of all angles between* 10″ *and* 90° *may be thus calculated.*

$$\text{Sin}\,(A+B) = 2\sin A \, . \cos B - \sin\,(A-B)\,;$$

and writing $n\,.\,10''$ for $\angle A$, and $10''$ for $\angle B$,

$$\text{Sin}\,(n+1)\,10'' = 2\sin n\,10''.\cos 10'' - \sin\,(n-1)\,10''.$$

$$\begin{aligned} \text{Now} \qquad 2\cos 10'' &= 1\cdot 9999999976496 \\ &= 2 - \cdot 0000000023504 \\ &= 2 - k \text{ suppose}\,; \end{aligned}$$

$$\begin{aligned} \therefore\ \text{Sin}\,(n+1)\,10'' &= 2\sin n\,10'' - k\,.\sin n\,10'' - \sin\,(n-1)\,10'' \\ &= \{\sin n\,.\,10'' - \sin\,(n-1)\,10''\} + \sin n\,10'' - k\,.\sin n\,10''. \end{aligned}$$

And by writing successively for n the numbers 1, 2, 3...

$$\text{Sin}\,20'' = (\sin 10'' - \sin\ \ 0'') + \sin 10'' - k\,.\sin 10'',$$

$$\text{Sin}\,30'' = (\sin 20'' - \sin 10'') + \sin 20'' - k\,.\sin 20'',$$

$$\text{Sin}\,40'' = (\sin 30'' - \sin 20'') + \sin 30'' - k\,.\sin 30''.$$

....

This method is not very laborious. In the last line, $\sin 30''$ and $(\sin 30'' - \sin 20'')$ are known from the two lines preceding, and the chief labour is in multiplying $\sin 30''$ by k.

The Sines of angles up to 60° having been successively calculated by the above method, those of angles between 60° and 90° may be thus determined.

$$\text{Sin}\,(60^\circ + A) - \sin\,(60^\circ - A) = 2\cos 60^\circ.\sin A = \sin A\,;$$

$$\therefore\ \text{Sin}\,(60^\circ + A) = \sin A + \sin\,(60^\circ - A).$$

So that if A be made to increase by 10″ at a time from 0° up to 60°, this formula will give by addition merely the sines of angles from 60° to 90°.

4. The Sines of angles up to 90° having been determined, their Cosines are also known.

For $\text{Cos}\,A = \sin\,(90^\circ - A)$.

Thus $\text{Cos}\,25^\circ = \sin\,(90^\circ - 25^\circ) = \sin 65^\circ$; $\text{Cos}\,72^\circ = \sin\,(90^\circ - 72^\circ) = \sin 18^\circ$; &c.

5. The Tangents, Cotangents, Secants, and Cosecants can be determined from the Sines and Cosines.

For $\text{Tan}\,A = \dfrac{\sin A}{\cos A}$, $\text{Cot}\,A = \dfrac{\cos A}{\sin A}$, $\text{Sec}\,A = \dfrac{1}{\cos A}$, $\text{Cosec}\,A = \dfrac{1}{\sin A}$.

8

6. Since,

$$\text{Sin}\, A = \cos(90^\circ - A), \quad \text{Tan}\, A = \cot(90^\circ - A), \quad \text{Sec}\, A = \text{cosec}\,(90^\circ - A),$$

the Sines, Tangents, and Secants of angles from 45° to 90° are the same respectively as the Cosines, Cotangents, and Cosecants of angles from 45° to 0°. Wherefore it is unnecessary to carry the tables further than to the angle 45°.

$$\begin{aligned} \text{Thus}\quad & \text{Cos}\, 72^\circ, 20' = \sin(90^\circ - 72^\circ, 20') = \sin 17^\circ, 40', \\ & \text{Sin}\, 72^\circ, 20' = \cos 17^\circ, 40', \quad \text{Tan}\, 72^\circ, 20' = \cot 17^\circ, 40', \\ & \text{Sec}\, 72^\circ, 20' = \text{cosec}\, 17^\circ, 40'. \end{aligned}$$

At the bottom of the page containing the Sines, &c. of angles from 17° to 18° is placed the angle 72°, and the column which at the top of the page is marked to indicate the Sines of angles from 17° to 18°, is marked at the bottom to shew the Cosines of angles from 72° to 73°; and so for the other Goniometric Ratios. See page 116.

7. *Formulæ of Verification.* Since the Goniometrical Ratios are determined *successively* from one another, one error will affect every successive result. As checks against the possibility of errors, several formulæ (*of verification* as they are called) are used to examine the accuracy of the results; and the values registered in the tables are presumed to be correct if they satisfy these formulæ.

The following are the principal formulæ of verification.

$$\left.\begin{array}{ll} (1) & \text{Sin}\, A = \frac{1}{2}\{\sqrt{1 + \sin 2A} - \sqrt{1 - \sin 2A}\}. \\ (2) & \text{Cos}\, A = \frac{1}{2}\{\sqrt{1 + \sin 2A} + \sqrt{1 - \sin 2A}\}. \end{array}\right\} \quad \ldots\ldots\ldots (41).$$

A being an angle less than 45°.

Again, $\text{Cos}\, 36^\circ = \dfrac{\sqrt{5}+1}{4}$; $\text{Cos}\, 72^\circ = \dfrac{\sqrt{5}-1}{4}$; Art. 58, (3), (2).

$$\left.\begin{aligned} \therefore\quad & \text{Sin}\,(36^\circ + A) - \sin(36^\circ - A) = 2\cos 36^\circ \,.\, \sin A = \frac{\sqrt{5}+1}{2} \,.\, \sin A, \\ \text{and}\quad & \text{Sin}\,(72^\circ + A) - \sin(72^\circ - A) = 2\cos 72^\circ \,.\, \sin A = \frac{\sqrt{5}-1}{2} \,.\, \sin A; \end{aligned}\right\}$$

∴ by subtraction,

$$\text{Sin}\,(36^\circ + A) + \sin(72^\circ - A) - \sin(36^\circ - A) - \sin(72^\circ + A) = \sin A;$$

(3). $\therefore\ \text{Sin}\, A + \sin(36^\circ - A) + \sin(72^\circ + A) = \sin(36^\circ + A) + \sin(72^\circ - A)$, which is *Euler's formula.*

Again, $\text{Sin}\,54^\circ = \frac{\sqrt{5}+1}{4}$; $\text{Sin}\,18^\circ = \frac{\sqrt{5}-1}{4}$; Art. 58, (3), (2).

$$\left.\begin{array}{l}\text{and } 2\sin 54^\circ . \cos A = \sin(54^\circ + A) + \sin(54^\circ - A) \\ \quad\;\; 2\sin 18^\circ . \cos A = \sin(18^\circ + A) + \sin(18^\circ - A)\end{array}\right\} .$$

(4). $\therefore$ $\text{Cos}\,A$, or $\text{Sin}\,(90^\circ - A)$,

$$= \sin(54^\circ + A) + \sin(54^\circ - A) - \sin(18^\circ + A) - \sin(18^\circ - A),$$

which is *Legendre's formula.*

This formula might have been proved by writing $90^\circ - A$ for A in Euler's formula.

Ex. To exemplify the use of these formulæ. By making $A = 13^\circ$ in *Legendre's* formula;

$$\text{Cos}\,13^\circ = \sin 67^\circ + \sin 41^\circ - \sin 31^\circ - \sin 5^\circ.$$

Now the tables give for the quantities in the second member of the equation 9205·049 + 6560·590 − 5150·381 − 871·557, which = 9743·701 the quantity given by the tables as the Cosine of 13°.

Since, therefore, these quantities satisfy the relation which ought to exist between the Sines of 67°, 41°, 31°, 5°, and the Cosine of 13°, it may be concluded, without much chance of error, that the values of these Goniometric Ratios are correctly given by the tables.

8. The values of the Goniometric Ratios having been thus calculated, multiplied by 10,000, verified, and registered in tables, are called "Tables of Natural Sines, Cosines, Tangents, Cotangents, Secants, and Cosecants." To find the *real* Goniometric Ratios from these tables, the tabular numbers have to be divided by 10,000; that is, the decimal point has to be removed four places to the left.

9. The *logarithmic* sines, cosines, tangents, &c. of angles will next be treated of, by which, rather than by the natural goniometric ratios, mathematical calculations are most frequently made.

A page from Sherwin's Logarithmic Tables, calculated for angles which differ from one another by 1′, is here subjoined, the Natural cosines, tangents, secants, and cosecants being omitted. The column following that of the N. sines, which is marked "$N . D . 1''$," will be explained in the next Appendix.

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17 Degrees.

M	N. Sine.	N. D. 1″.	L. Sine.	Diff. 1 sec.	Co-secants.	L. Tan.	Diff. 1 sec.	L. Sec.	Diff. 1 sec.	Co-sines.	M
0	2923·717		9·4659353		10·5340647	9·4853390		10·0194037		9·9805963	60
1	2926·499		9·4663483	68·833	10·5336517	9·4857907	75·283	10·0194423	6·433	9·9805577	59
2	2929·280	46·36	9·4667609	68·766	10·5332391	9·4862419	75·200	10·0194810	6·450	9·9805190	58
3	2932·061		9·4671730	68·683	10·5328270	9·4866928	75·150	10·0195197	6·450	9·9804803	57
4	2934·842		9·4675848	68·633	10·5324152	9·4871433	75·083	10·0195585	6·466	9·9804415	56
5	2937·623		9·4679960	68·533	10·5320040	9·4875933	75·000	10·0195973	6·466	9·9804027	55
6	2940·403		9·4684069	68·483	10·5315931	9·4880430	74·950	10·0196361	6·466	9·9803639	54
7	2943·183	46·33	9·4688173	68·400	10·5311827	9·4884924	74·900	10·0196750	6·483	9·9803250	53
8	2945·963		9·4692273	68·333	10·5307727	9·4889413	74·816	10·0197140	6·500	9·9802860	52
9	2948·743		9·4696369	68·266	10·5303631	9·4893898	74·750	10·0197529	6·500	9·9802471	51
10	2951·522		9·4700461	68·200	10·5299539	9·4898380	74·700	10·0197919	6·500	9·9802081	50
11	2954·302		9·4704548	68·116	10·5295452	9·4902858	74·633	10·0198310	6·516	9·9801690	49
12	2957·081	46·31	9·4708631	68·050	10·5291369	9·4907332	74·566	10·0198701	6·516	9·9801299	48
13	2959·859		9·4712710	67·983	10·5287290	9·4911802	74·616	10·0199092	6·516	9·9800908	47
14	2962·638		9·4716785	67·916	10·5283215	9·4916269	74·450	10·0199484	6·533	9·9800516	46
15	2965·416		9·4720856	67·850	10·5279144	9·4920731	74·366	10·0199876	6·533	9·9800124	45
16	2968·194		9·4724922	67·766	10·5275078	9·4925190	74·316	10·0200268	6·533	9·9799732	44
17	2970·971	46·29	9·4728985	67·716	10·5271015	9·4929646	74·260	10·0200661	6·550	9·9799339	43
18	2973·749		9·4733043	67·633	10·5266957	9·4934097	74·183	10·0201054	6·550	9·9798946	42
19	2976·526		9·4737097	67·566	10·5262903	9·4938545	74·133	10·0201448	6·566	9·9798552	41
20	2979·303		9·4741146	67·483	10·5258854	9·4942988	74·050	10·0201842	6·566	9·9798158	40
21	2982·079		9·4745192	67·433	10·5254808	9·4947429	74·016	10·0202236	6·566	9·9797764	39
22	2984·856	46·28	9·4749234	67·366	10·5250766	9·4951865	74·060	10·0202631	6·583	9·9797369	38
23	2987·632		9·4753271	67·283	10·5246729	9·4956298	73·883	10·0203027	6·600	9·9796973	37
24	2990·408		9·4757304	67·216	10·5242696	9·4960727	73·816	10·0203422	6·600	9·9796578	36
25	2993·184		9·4761334	67·166	10·5238666	9·4965152	73·416	10·0203818	6·600	9·9796182	35
26	2995·959		9·4765359	67·083	10·5234641	9·4969574	73·700	10·0204215	6·616	9·9795785	34
27	2998·734	46·25	9·4769380	67·016	10·5230620	9·4973991	73·616	10·0204612	6·616	9·9795388	33
28	3001·509		9·4773396	66·933	10·5226604	9·4978406	73·583	10·0205009	6·616	9·9794991	32
29	3004·284		9·4777409	66·883	10·5222591	9·4982816	73·500	10·0205407	6·633	9·9794593	31
30	3007·058		9·4781418	66·816	10·5218582	9·4987223	73·450	10·0205805	6·633	9·9794195	30
31	3009·832		9·4785423	66·750	10·5214577	9·4991626	73·383	10·0206204	6·648	9·9793796	29
32	3012·606	46·23	9·4789423	66·666	10·5210577	9·4996026	73·333	10·0206602	6·633	9·9793398	28
33	3015·380		9·4793420	66·616	10·5206580	9·5000422	73·266	10·0207002	6·666	9·9792998	27
34	3018·153		9·4797412	66·533	10·5202588	9·5004814	73·200	10·0207401	6·650	9·9792599	26
35	3020·926		9·4801401	66·483	10·5198599	9·5009203	73·150	10·0207802	6·683	9·9792198	25
36	3023·699		9·4805385	66·400	10·5194615	9·5013588	73·083	10·0208202	6·666	9·9791798	24
37	3026·471	46·22	9·4809366	66·350	10·5190634	9·5017969	73·016	10·0208603	6·683	9·9791397	23
38	3029·244		9·4813342	66·266	10·5186658	9·5022347	72·966	10·0209004	6·683	9·9790996	22
39	3032·016		9·4817315	66·216	10·5182685	9·5026721	72·900	10·0209406	6·700	9·9790594	21
40	3034·788		9·4821283	66·133	10·5178717	9·5031092	72·850	10·0209808	6·700	9·9790192	20
41	3037·559		9·4825248	66·083	10·5174752	9·5035459	72·783	10·0210211	6·716	9·9789789	19
42	3040·331	46·20	9·4829208	66·000	10·5170792	9·5039822	72·716	10·0210614	6·716	9·9789386	18
43	3043·102		9·4833165	65·950	10·5166835	9·5044182	72·666	10·0211017	6·716	9·9788983	17
44	3045·872		9·4837117	65·866	10·5162883	9·5048538	72·600	10·0211421	6·733	9·9788579	16
45	3048·643		9·4841066	65·816	10·5158934	9·5052891	72·550	10·0211825	6·733	9·9788175	15
46	3051·413		9·4845010	65·733	10·5154990	9·5057240	72·483	10·0212230	6·750	9·9787770	14
47	3054·183	46·16	9·4848951	65·683	10·5151049	9·5061586	72·433	10·0212635	6·750	9·9787365	13
48	3056·953		9·4852888	65·616	10·5147112	9·5065928	72·366	10·0213040	6·750	9·9786960	12
49	3059·723		9·4856820	65·533	10·5143180	9·5070267	72·316	10·0213446	6·766	9·9786554	11
50	3062·492		9·4860749	65·483	10·5139251	9·5074602	72·250	10·0213852	6·766	9·9786148	10
51	3065·261		9·4864674	65·416	10·5135326	9·5078933	72·183	10·0214259	6·783	9·9785741	9
52	3068·030	46·15	9·4868595	65·350	10·5131405	9·5083261	72·133	10·0214666	6·783	9·9785334	8
53	3070·798		9·4872512	65·283	10·5127488	9·5087586	72·083	10·0215073	6·783	9·9784927	7
54	3073·566		9·4876426	65·233	10·5123574	9·5091907	72·016	10·0215481	6·800	9·9784519	6
55	3076·334		9·4880335	65·150	10·5119665	9·5096224	71·950	10·0215889	6·800	9·9784111	5
56	3079·102		9·4884240	65·083	10·5115760	9·5100539	71·916	10·0216298	6·816	9·9783702	4
57	3081·869	46·11	9·4888142	65·003	10·5111858	9·5104849	71·833	10·0216707	6·816	9·9783293	3
58	3084·636		9·4892040	64·966	10·5107960	9·5109156	71·783	10·0217117	6·833	9·9782883	2
59	3087·403		9·4895934	64·900	10·5104066	9·5113460	71·733	10·0217526	6·816	9·9782474	1
60	3090·170		9·4899824	64·833	10·5100176	9·5117760	71·666	10·0217937	6·850	9·9782063	0
M	Co-sines.				L. Sec.	Co-tans.		Co-secants.		L. Sine.	M

72 Degrees.

APPENDIX III.

ON THE LOGARITHMIC TABLES OF GONIOMETRIC RATIOS.

1. **When** the Sines, Cosines, &c., of angles have been determined, their logarithms may be found from the tables of the logarithms of numbers. There are, however, methods by which the logarithms of the Goniometric Ratios can be found independently.

2. *To find* $l_{10}\sin\theta$, $\sin\theta$ *not being given.*

$$\text{Sin}\,\theta = \theta \,.\left(1-\frac{\theta^2}{\pi^2}\right).\left(1-\frac{\theta^2}{2^2.\pi^2}\right).\left(1-\frac{\theta^2}{3^2.\pi^2}\right)\ldots\ldots \text{Art. } 133.$$

By making $\theta = \frac{m}{n}\,.\,\frac{\pi}{2}$, and taking the logarithms of both sides of this equation,

$$\therefore\; l_{10}\sin\left(\frac{m}{n}\,.\,\frac{\pi}{2}\right) = l_{10}\left(\frac{m}{n}\,.\,\frac{\pi}{2}\right) + l_{10}\left(1-\frac{m^2}{2^2n^2}\right) + l_{10}\left(1-\frac{m^2}{4^2n^2}\right) + l_{10}\left(1-\frac{m^2}{6^2n^2}\right) + \ldots$$

and $l_{10}\left(1-\frac{m^2}{4^2n^2}\right)$, $l_{10}\left(1-\frac{m^2}{6^2n^2}\right)$, &c. being expanded by (ii.), p. 100.

$$= l_{10}\left(\frac{m}{n}\,.\,\frac{\pi}{2}\right) + l_{10}\left(\frac{2^2n^2-m^2}{2^2n^2}\right)$$

$$-\frac{1}{l_e 10}\,.\left(\frac{m^2}{4^2n^2} + \frac{1}{2}\,.\,\frac{m^4}{4^4n^4} + \frac{1}{3}\,.\,\frac{m^6}{4^6n^6} + \ldots\right)$$

$$-\frac{1}{l_e 10}\,.\left(\frac{m^2}{6^2n^2} + \frac{1}{2}\,.\,\frac{m^4}{6^4n^4} + \frac{1}{3}\,.\,\frac{m^6}{6^6n^6} + \ldots\right)$$

$$-\frac{1}{l_e 10}\,.\left(\frac{m^2}{8^2n^2} + \frac{1}{2}\,.\,\frac{m^4}{8^4n^4} + \frac{1}{3}\,.\,\frac{m^6}{8^6n^6} + \ldots\right)$$

$$- \&\text{c}.$$

Now $l_{10}\left(\frac{m}{n}\,.\,\frac{\pi}{2}\right) = l_{10}m + l_{10}\pi - l_{10}n - l_{10}2,$

and $l_{10}\frac{2^2n^2-m^2}{2^2n^2} = l_{10}\{(2n+m).(2n-m)\} - l_{10}(2^2n^2)$

$$= l_{10}(2n+m) + l_{10}(2n-m) - 2.\{l_{10}2 + l_{10}n\};$$

$$\therefore \quad \mathrm{l}_{10}\sin\left(\frac{m}{n}\cdot\frac{\pi}{2}\right)=\mathrm{l}_{10}m+\mathrm{l}_{10}(2n+m)+\mathrm{l}_{10}(2n-m)+\mathrm{l}_{10}\pi-3.\{\mathrm{l}_{10}n+\mathrm{l}_{10}2\}$$

$$-\frac{1}{\mathrm{l}_e 10}\times\left\{\begin{array}{l}\left(\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\dots\right)\frac{m^2}{n^2}\\ +\frac{1}{2}\left(\frac{1}{4^4}+\frac{1}{6^4}+\frac{1}{8^4}+\dots\right)\frac{m^4}{n^4}\\ +\frac{1}{3}\left(\frac{1}{4^6}+\frac{1}{6^6}+\frac{1}{8^6}+\dots\right)\frac{m^6}{n^6}\\ +\,\&\text{c.}\end{array}\right\}.$$

By giving m and n different values, the logarithmic sines of all angles may be found by this formula.

3. In like manner from the series

$$\operatorname{Cos}\theta=\left(1-\frac{2^2\theta^2}{\pi^2}\right).\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right).\left(1-\frac{2^2\theta^2}{5^2\pi^2}\right)\dots\dots \text{ Art. 133.}$$

the following formula is obtained;

$$\mathrm{l}_{10}\cos\left(\frac{m}{n}\cdot\frac{\pi}{2}\right)=\mathrm{l}_{10}(n+m)+\mathrm{l}_{10}(n-m)-2\,\mathrm{l}_{10}n$$

$$-\frac{1}{\mathrm{l}_e 10}\times\left\{\begin{array}{l}\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots\right)\frac{m^2}{n^2}\\ +\frac{1}{2}\left(\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\dots\right)\frac{m^4}{n^4}\\ +\frac{1}{3}\left(\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\dots\right)\frac{m^6}{n^6}\\ +\,\&\text{c.}\end{array}\right\}.$$

4. The logarithms of the Sines and Cosines having been thus determined, those of

$$\operatorname{Tan}\theta=\frac{\sin\theta}{\cos\theta},\quad \operatorname{Sec}\theta=\frac{1}{\cos\theta},\quad \operatorname{Cot}\theta=\frac{\cos\theta}{\sin\theta},\quad \operatorname{Cosec}\theta=\frac{1}{\sin\theta},$$

may be severally found.

5. Since all Sines and Cosines are, generally, less than 1, their logarithms to base 10 are negative. In order to avoid the inconvenience of printing negative characteristics, the logarithms to base of 10 of *all* Goniometrical Ratios have been increased by 10, and the resulting numbers being registered are *called* "Tables of Logarithmic Sines, Cosines, &c."

Hence if any one of these *tabular* logarithmic quantities be given, by subtracting 10 from it, the *real* logarithm of the goniometric ratio may be obtained.

These tabular logarithmic quantities will be indicated by the letter L; thus the tabular logarithmic sine of A, or $10 + l_{10}\sin A$, will be written $L\sin A$.

6. The common Logarithmic Tables of Goniometric Ratios are calculated for angles which differ from one another by one minute. If, besides degrees and minutes, the angle contain some seconds, its tabular logarithmic function may, with certain exceptions, be found on the principle proved in the next Article.

7. *The increments of tabular logarithmic sines, &c., of angles vary, except in certain cases, as the increment of the angle.*

Let the angle A receive the increments a'', and $60''$, successively.

Then $\operatorname{Sin}(A+a'') = \sin A\,.\left\{1 + \dfrac{\sin(A+a'') - \sin A}{\sin A}\right\}$

$= \sin A\,.\left\{1 + \dfrac{\cos A\,.\sin a''}{\sin A}\right\}$, unless $A = 90^\circ$ nearly; Art. 59. Cor.

$$\therefore\ l_{10}\sin(A+a'') = l_{10}\sin A + l_{10}(1 + \cot A\,.\sin a'');$$

$$\therefore\ \{10 + l_{10}\sin(A+a'')\} - \{10 + l_{10}\sin A\} = l_{10}(1 + \cot A\,.\sin a'');$$

$$\therefore\ L\sin(A+a'') - L\sin A$$

$= \dfrac{1}{l_e 10}\,.\{\cot A\,.\sin a'' - \tfrac{1}{2}\cot^2 A\,.\sin^2 a'' + \dots\}$ App. I. (14) (ii.)

$= \dfrac{1}{l_e 10}\,.\cot A\,.\sin a''$; by neglecting the higher powers of $\cot A\,.\sin a''$;

which may be done unless $A = 2n\,.90^\circ$ nearly.

And writing $60''$ for a'' in this equation, it becomes

$$L\sin(A+60'') - L\sin A = \frac{1}{l_e 10}\,.\cot A\,.\sin 60'';$$

$$\therefore\ \frac{L\sin(A+a'') - L\sin A}{L\sin(A+60'') - L\sin A} = \frac{\sin a''}{\sin 60''} = \frac{a}{60}. \quad \text{Art. 106. Cor.}$$

And in an exactly similar manner it may be shewn that for the Cosine, Tangent, &c., of an angle, the increment of the tabular logarithm varies as the increment of the angle, except in those cases mentioned in the Corollaries to Arts. 60, 61, 62.

8. *To explain the meaning and use of the columns of differences for one second* (Diff. 1″), *which are placed after the columns of logarithmic sines, tangents, &c.*

If α'' become 1″, the equation arrived at in the last Article becomes,

$$L\sin(A+1'') - L\sin A = \{\,L(\sin A + 60'') - L\sin A\,\}\,.\,\frac{1}{60},$$

which is the difference for $L\sin A$ corresponding to one second.

Now, if this quantity be computed and registered, $L(\sin A + \alpha'') - L\sin A$ may be determined, when α is given, by merely multiplying this registered difference by α; and when $L\sin(A+\alpha'')$ is given, α may be found by dividing $L\sin(A+\alpha'') - L\sin A$ by this difference. For

$$\begin{aligned} L\sin(A+\alpha'') - L\sin A &= \{\,L\sin(A+60'') - L\sin A\,\}\,.\,\frac{\alpha}{60} \\ &= \frac{L\sin(A+60'') - L\sin A}{60}\,.\,\alpha, \end{aligned}$$

$$\text{and } \alpha = \frac{L\sin(A+\alpha'') - L\sin A}{\frac{1}{60}\,.\,\{\,L\sin(A+60'') - L\sin A\,\}}.$$

$$\begin{aligned} \text{Thus, }\quad L\sin 17^\circ,\ 1' &= 9{\cdot}4663483 \\ L\sin 17^\circ,\ \ &= 9{\cdot}4659353 \\ \hline \therefore\ L\sin 17^\circ,\ 1' - L\sin 17^\circ &= {\cdot}0004130 \end{aligned}$$

Now $\frac{4130}{60} = 68{\cdot}833$, which is the quantity put down in the Tables as the difference for one second to L sines of angles between 17° and 17°, 1′.

The significant part of the difference is considered as a whole number, or the *real* difference is multiplied by 10^7, in order to avoid the necessity of printing the three or four cyphers which in nearly every case precede the significant part of the difference*.

9. Ex. 1. *To find* L sin 17°, 14′, 12″.

$$\begin{aligned} L\sin 17^\circ,\ 14', \quad &= 9{\cdot}4716785 \\ \text{Now Diff. for } 1'' = 67{\cdot}850 \quad & \\ \therefore\ \text{Diff. for } 12'' = 814{\cdot}200 = \quad & \phantom{9{\cdot}471}814{\cdot}2 \\ \hline \therefore\ L\sin 17^\circ,\ 14',\ 12'' &= 9{\cdot}4717599. \end{aligned}$$

---

* The column of differences for the natural sines, &c. of angles are computed after a manner similar to this, and the differences themselves are all multiplied by 1000, for the sake of avoiding the necessity of printing the cyphers immediately following the decimal point.

Ex. 2. *If* L sin A = 9·4685537, *required* A.

$$L \sin A = 9{\cdot}4685537$$
$$L \sin 17^\circ, 6' = 9{\cdot}4684069$$
$$\text{Diff.} = 1468$$

Now *Diff. for* 1″ is in this case 68·400, and $\frac{1468}{68{\cdot}400} = 21{\cdot}46$; $\therefore A = 17^\circ, 6', 21''{\cdot}46$.

Ex. 3. *If* L cos A = 9·9784328, *required* A.

In this case, because the increase of the angle is attended by the decrease of the *L* cosine, Art. 60, Cor. 2, the given *L* cosine must be *subtracted* from that in the tables which is next greater than it.

$$\text{Now } L \cos 17^\circ, 54' = 9{\cdot}9784519$$
$$L \cos A = 9{\cdot}9784328$$
$$\text{Diff.} = 191$$

Now *Diff. for* 1″ in this case is 6·800, and $\frac{191}{6{\cdot}800} = 28{\cdot}088 = 28{\cdot}09$ nearly; and the angle required is $17^\circ, 54', 28''{\cdot}09$.

Ex. 4. *Required the* L *cosine of* 72°, 5′, 8″.

By the Tables, p. 116,

$$L \cos 72^\circ, 5' = 9{\cdot}4880335, \text{ and } Diff. \text{ for } 1'' = 65{\cdot}15;$$
$$\therefore \text{Diff. for } 8'' = 521{\cdot}2$$
$$\therefore L \cos 72^\circ, 5', 8'' = 9{\cdot}4879814$$

The difference for the additional seconds being in this case *subtracted* from *L* cos 72°, 5′. Art. 60.

Note. It may here be observed, that the difference for additional seconds must be *added* for *L* sines, *L* tangents, and *L* secants, Arts. 59, 61, 62; and *subtracted* for *L* cosines, Art. 60, *L* cotangents, and *L* cosecants.

10. *To shew that the same columns of "Differences for* 1″" *serve for* L sin A *and* L cosec A, *for* L cos A *and* L sec A, *and for* L tan A *and* L cot A.

$$\text{For } \operatorname{Sin} A = \frac{1}{\operatorname{cosec} A}; \quad \therefore l_{10} \sin A = -l_{10} \operatorname{cosec} A,$$

$$\therefore L \sin A = 10 + l_{10} \sin A = 20 - (10 + l_{10} \operatorname{cosec} A) = 20 - L \operatorname{cosec} A.$$

$$\text{Similarly,} \quad L \sin (A + 1'') = 20 - L \operatorname{cosec} (A + 1'');$$

$$\therefore L \sin(A + 1'') - L \sin A = -\{ L \operatorname{cosec}(A + 1'') - L \operatorname{cosec} A \}.$$

Hence a column of "differences for 1″" is printed *between* the columns of logarithmic sines and cosecants; serving to the former as a column of *increments* for 1″, and to the latter as a column of *decrements* for 1″.

In like manner it may be shewn that

$$L\cos(A+1'') - L\cos A = -\{L\sec(A+1'') - L\sec A\}$$

$$L\tan(A+1'') - L\tan A = -\{L\cot(A+1'') - L\cot A\}.$$

Wherefore the columns of cosines and secants have the same differences for 1″, as also have the tangents and cotangents: and it is to be observed that these differences serve respectively as increments to the secants, (Art. 61.), and to the tangents (Art. 62.), and as decrements to the cosines (Art. 60.), and to the cotangents.

11. Before the increment of the tabular logarithm of a Goniometrical Ratio can be determined from the *small* given increment of the angle, or conversely, the two following conditions must be fulfilled.

I. The logarithmic increment must in that particular case vary as the increment of the angle.

II. The increment of the logarithm must not be an exceedingly small quantity.

Thus, if it were required to determine L sin 89°, 40′, 3″, from Tables in which L sines were registered for all angles from 0° to 90° which differed from one another by 1′, it would be found that,

$$\begin{aligned} L\sin 89^\circ,41' &= 9{\cdot}9999934 \\ L\sin 89^\circ,40' &= 9{\cdot}9999927 \\ \hline \therefore\ \text{Difference for } 60'' &= \qquad\quad 7 \end{aligned}$$

Wherefore, if even the first of these conditions held for the L sines of angles about 89°, 40′ in magnitude, (which it does not, Art. 59.), yet a difference of 1 in the L sine would produce a difference of $\frac{60''}{7}$, or 9″ nearly, in the angle; and therefore any increment of 89°, 40′ which was not greater than 8″, would produce no change at all in the first seven figures following the decimal point of L sin 89°, 40′.

12. *To determine the degree of accuracy to which additional seconds may be calculated in a given case by making use of the principle assumed in the text.*

Let n be the number of seconds by which a difference of unity is produced in a certain logarithmic function of a given angle, and let l be the difference for 60″;—the differences being considered in both cases as whole numbers;—

$$\text{Then } \frac{n}{60} = \frac{1}{l}, \quad \text{and } n = \frac{60}{l}.$$

The quantity $\frac{60}{l}$ therefore gives the number of additional seconds corresponding to the least possible increase of the given logarithmic function of the angle, and is consequently the measure of the degree of accuracy to which small increments of the angle may be calculated on the principle that the Increments of the angle vary as the Increments of the logarithm of some particular Goniometric function of it.

13. It has been observed, App. III. 5, that the *real* logarithm of a Goniometrical quantity is obtained by subtracting 10 from the tabular logarithm. It is therefore necessary,

To establish a general rule for supplying the TENS *when the tabular logarithms of goniometrical quantities are used.*

Let $\operatorname{Cos}^n A = a \,.\, \sin^m B \,.\, \tan^p C$ be any trigonometrical formula adapted to logarithmic computation, App. I. 25; B and C being given angles, and A being required.

$$\text{Then } \; n \,.\, \mathrm{l}_{10} \cos A = \mathrm{l}_{10} a + m \,.\, \mathrm{l}_{10} \sin B + p \,.\, \mathrm{l}_{10} \tan C\,;$$

$$\therefore\; n \,.\, (10 + \mathrm{l}_{10} \cos A) - n \,.\, 10 = \mathrm{l}_{10} a + m \,.\, (10 + \mathrm{l}_{10} \sin B) - m \,.\, 10$$

$$+ p \,.\, (10 + \mathrm{l}_{10} \tan C) - p \,.\, 10\,;$$

$$\therefore\; n \,.\, L \cos A = \mathrm{l}_{10} a + m \,.\, L \sin B + p \,.\, L \tan C + \{n - (m + p)\} \times 10.$$

And here n, m, p may be whole or fractional.

Whence the rule; *Add to the second member of the equation as many tens as the number of times the tabular logarithms of the goniometrical ratios have been taken in the former member of the equation exceeds the number of times they have been taken in the latter member.*

Ex. 1. $\text{Tan}^5 A = \cos B \, . \cos^2 C$;

$$\therefore\ 5 \, . L \tan A = L \cos B + 2 \, . L \cos C + \{5 - (1+2)\} \, . 10 = L \cos B + 2 \, . L \cos C + 20.$$

Ex. 2. $\text{Tan}^2 A \, . \sin^6 B = \frac{a}{b} \, . \sec^4 C$,

$$2 \, . L \tan A + 6 \, . L \sin B = l_{10} a - l_{10} b + 4 \, . L \sec C + \{(2+6) - 4\} \, . 10$$
$$= l_{10} a - l_{10} b + 4 \, . L \sec C + 40.$$

Ex. 3. $\text{Tan}^{\frac{3}{2}} A = \dfrac{2 \, . \sin^{\frac{5}{2}} B}{\cos^{\frac{2}{3}} C}$;

$$\therefore\ \frac{3}{2} \, . L \tan A = l_{10} 2 + \frac{5}{2} \, . L \sin B - \frac{2}{3} \, . L \cos C + \left\{ \frac{3}{2} - \left(\frac{5}{2} - \frac{2}{3} \right) \right\} . 10$$
$$= l_{10} 2 + \frac{5}{2} \, . L \sin B - \frac{2}{3} \, . L \cos C - \tfrac{1}{3} \, . 10.$$

Had both sides of this equation been raised to the sixth power, the fractional indices would have disappeared, and the value of $\tan A$ would, practically, have been determined much more easily.

14. Lastly, the methods will be explained by which small angles are determined from their L sines, and conversely.

When an angle receives a small increment, the Differential Calculus affords the means of determining with facility the consequent increase of the Goniometric Ratio.

Required the increase of L $\sin \theta$ *arising from* θ *receiving a small increment* $\delta\theta$.

By TAYLOR'S THEOREM,

$$L \sin(\theta + \delta\theta) = L \sin\theta + d_\theta L \sin\theta \, . \, \delta\theta + d_\theta^2 \, L \sin\theta \, . \frac{(\delta\theta)^2}{1 \, . \, 2} + \ldots$$

$$\text{Now} \quad L \sin\theta = 10 + l_{10} \sin\theta\,;$$

$$\therefore\ d_\theta \, L \sin\theta = \frac{1}{l_\epsilon 10} \, . \frac{\cos\theta}{\sin\theta} = \frac{1}{l_\epsilon 10} \, . \cot\theta,$$

$$\text{whence,} \quad d_\theta^2 \, L \sin\theta = - \frac{1}{l_\epsilon 10} \, . \operatorname{cosec}^2\theta\,;$$

$$\therefore\ L \sin(\theta + \delta\theta) - L \sin\theta = \frac{1}{l_\epsilon 10} \, . \cot\theta \, . \, \delta\theta - \frac{1}{l_\epsilon 10} \, . \operatorname{cosec}^2\theta \, . \frac{(\delta\theta)^2}{1 \, . \, 2} + \ldots\ldots$$

15. Now if θ be very small, cosec θ is very large, and therefore unless $\delta\theta$ be exceedingly small also, the second difference, $-\frac{1}{l_\epsilon 10} \cdot \operatorname{cosec}^2\theta \cdot \frac{(\delta\theta)^2}{1 \cdot 2}$, is of such a magnitude that it cannot be neglected in comparison with the first difference, $\frac{1}{l_\epsilon 10} \cdot \cot\theta \cdot \delta\theta$. In this case, therefore, since the increment of $L \sin\theta$ does not vary as $\delta\theta$, the simple power of the increment of θ, the quantity to be added to $L \sin\theta$ for a small increment of θ cannot be obtained by a simple proportion, but will have to be determined approximately by the tedious process of computing the first two terms of the series which gives the value of the increment of $L \sin\theta$.

There are three methods of escaping this inconvenience.

16. *The first method* is, to construct tables for the first few degrees to intervals of a second, instead of to intervals of a minute as the tables in ordinary use are constructed.

Here $\delta\theta$ must be less than one second, and $L \sin(\theta + \delta\theta)$ may be roughly computed to decimal parts of a second by neglecting the second term of series of (14), p. 124; in which case,

$$L \sin(\theta + \delta\theta) - L \sin\theta = \frac{1}{l_\epsilon 10} \cdot \cot\theta \cdot \delta\theta\,;$$

Therefore, for any particular value of θ, $\Delta L \sin\theta \propto \delta\theta$.

17. *Second method.* By the following formulæ which are given by Maskelyne in his introduction to Taylor's Logarithms, a small angle may be determined very accurately to decimal parts of a second from its L sine and conversely, by the aid of tables of L sines and L cosines which are calculated to every second for a few degrees.

$$\operatorname{Sin}\theta = \theta - \frac{\theta^3}{2 \cdot 3} + \ldots \qquad = \theta - \frac{\theta^3}{2 \cdot 3} \text{ nearly.}$$

$$\operatorname{Cos}\theta = 1 - \frac{\theta^2}{1 \cdot 2} + \ldots \qquad = 1 - \frac{\theta^2}{2} \quad \text{nearly.}$$

$$\therefore \frac{\operatorname{Sin}\theta}{\theta} = 1 - \frac{\theta^2}{2 \cdot 3} = \left(1 - \frac{\theta^2}{2}\right)^{\frac{1}{3}} \text{ nearly, } = \cos^{\frac{1}{3}}\theta.$$

Let θ be an angle containing n'', then $n = \frac{\theta}{\sin 1''}$, or $\theta = n \cdot \sin 1''$;

$$\text{and } \therefore \operatorname{Sin}\theta = n \cdot \sin 1'' \cdot \cos^{\frac{1}{3}}\theta,$$

Writing n'' for θ, and taking the logarithms,

$$L \sin n'' = l_{10} n + L \sin 1'' + \tfrac{1}{3} L \cos n'' - \tfrac{1}{3} \, . \, 10;$$

$$\therefore \quad L \sin n'' = l_{10} n + L \sin 1'' - \tfrac{1}{3} \, . \, (10 - L \cos n'') \ldots\ldots\ldots\ldots\ldots\ldots \text{(i.)},$$

$$l_{10} n = L \sin n'' + \tfrac{1}{3} (10 - L \cos n'') - L \sin 1'' \ldots\ldots\ldots\ldots \text{(ii.)}$$

Def. The quantity $10 - L \cos n''$ is called "the arithmetic complement" of $L \cos n''$.

18. It is to be remarked, that in using these formulæ to determine $L \sin n''$ when n is given, or conversely, an *approximate* value of $L \cos n''$ may be taken from the tables and written in the second member of the equations without sensibly affecting the result, because the variation of $L \cos n''$ is exceedingly small when n is small, as may easily be shewn*. The whole matter will be rendered more clear by an example.

Ex. *If* L sin n″ = 7·3217783, *required* n.

Now Taylor's tables give,

$L \sin 7', 12''$	7·3210583	$L \cos 7', 12''$	9·9999990
$L \sin 7', 13''$	7·3220624	$L \cos 7', 13''$	9·9999990

Therefore the angle is $7', 12''$ nearly; and $7', 12''$ is the approximate value of the angle which must be taken in the second member of (ii.) of the last Article.

Now by (ii.),

$$\begin{array}{rl} L \sin n'' = & 7{\cdot}3217783 \\ \left.\begin{array}{l} \tfrac{1}{3}(10 - L \cos 7', 12'') \\ \text{or } \tfrac{1}{3} \times {\cdot}0000010, \end{array}\right\} = & \underline{{\cdot}0000003} \\ & 7{\cdot}3217786 \\ L \sin 1'' = & \underline{4{\cdot}6855749} \end{array}$$

$$\therefore \; l_{10} n = 2{\cdot}6362037 = l_{10} 432{\cdot}717;$$

$\therefore n = 432{\cdot}717$, or the angle required is $7', 12''{\cdot}717$.

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* $\cos \theta = 1 - \dfrac{\theta^2}{1 \, . \, 2} + \dfrac{\theta^4}{1 \, . \, 2 \, . \, 3 \, . \, 4} \ldots\ldots = 1 - \dfrac{\theta^2}{2}$ nearly;

$\therefore \; l_{10} \cos \theta$, or $L \cos \theta - 10$, $= - \dfrac{1}{l_\epsilon 10} \, . \left\{ \dfrac{\theta^2}{2} + \tfrac{1}{2} \, . \left(\dfrac{\theta^2}{2} \right)^2 + \ldots \right\}$;

$\therefore \; d_\theta (L \cos \theta - 10) = - \dfrac{1}{l_\epsilon 10} \, . \, (\theta + \dfrac{\theta^3}{2} + \ldots)$, a very small quantity, if θ be an angle of a few minutes only.

19. In like manner a formula may be established for finding $L \tan n''$ from n, and conversely.

$$\text{Tan}\,\theta = \frac{\sin\theta}{\cos\theta} = \frac{\theta - \frac{\theta^3}{2.3}}{1 - \frac{\theta^2}{2}} \text{ nearly;}$$

$$\therefore \frac{\tan\theta}{\theta} = \frac{1 - \frac{\theta^2}{2.3}}{1 - \frac{\theta^2}{2}} = \frac{\left(1 - \frac{\theta^2}{2}\right)^{\frac{1}{3}}}{1 - \frac{\theta^2}{2}} = \left(1 - \frac{\theta^2}{2}\right)^{-\frac{2}{3}} = (\cos\theta)^{-\frac{2}{3}};$$

$$\therefore L \frac{\tan n''}{n \,.\, \sin 1''} = (\cos n'')^{-\frac{2}{3}};$$

$$\therefore L \tan n'' = \text{l}_{10} n + L \sin 1'' - \frac{2}{3} \,.\, L \cos n'' + \frac{2}{3} \,.\, 10;$$

$$\therefore L \tan n'' = \text{l}_{10} n + L \sin 1'' + \frac{2}{3} \,.\, (10 - L \cos n'') \text{.......................} \text{(iii.)}$$

$$\text{And } \text{l}_{10} n = L \tan n'' - L \sin 1'' - \frac{2}{3} (10 - L \cos n'') \text{.....................} \text{(iv.)}$$

20. *Delambre's Tables. The third method* alluded to, (15, p. 125,) is to construct tables as far as an angle of one degree, which give $\text{l}_{10} \frac{\sin\theta}{\theta}$, (or tables which give $\text{l}_{10} \frac{\sin\theta}{\theta} + L \sin 1''$), for every second.

Such tables are, perhaps, printed in no collection except those of Callet; they may, however, be easily constructed for use in the following manner:

Let θ be an angle of n seconds; $\therefore\ \theta = n \,.\, \sin 1''$.

$$\text{Then } \text{l}_{10} \frac{\sin\theta}{\theta} = \text{l}_{10} \frac{\sin n''}{n \,.\, \sin 1''} = \text{l}_{10} \sin n'' - \text{l}_{10} n - \text{l}_{10} \sin 1''$$

$$= L \sin n'' - \text{l}_{10} n - L \sin 1'';$$

$$\therefore \text{l}_{10} \frac{\sin\theta}{\theta} + L \sin 1'' = L \sin n'' - \text{l}_{10} n.$$

Similarly, if θ be an angle of n minutes,

$$\text{l}_{10} \frac{\sin\theta}{\theta} + L \sin 1' = L \sin n' - \text{l}_{10} n.$$

21. *To determine the Sine of a given small angle, or conversely, from Delambre's Tables.*

Since $\operatorname{Sin}\theta = \frac{\sin\theta}{\theta}\,.\,\theta = \frac{\sin\theta}{\theta}\,.\,n\,.\sin 1''$;

$$\therefore\ L\sin\theta = (\mathrm{l}_{10}\frac{\sin\theta}{\theta} + L\sin 1'') + \mathrm{l}_{10}n\,;$$

$$\therefore\ L\sin n'' = (\mathrm{l}_{10}\frac{\sin\theta}{\theta} + L\sin 1'') + \mathrm{l}_{10}n \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i.)}$$

$$\text{And}\quad \mathrm{l}_{10}n = L\sin n'' - (\mathrm{l}_{10}\frac{\sin\theta}{\theta} + L\sin 1'') \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii.)}$$

The most convenient tables are evidently those which give $(\mathrm{l}_{10}\frac{\sin\theta}{\theta} + L\sin 1'')$ for every second.

22. Ex. 1. *To determine* L sin n″ *by Dalambre's Tables.*

Since, as shewn below*, $\mathrm{l}_{10}\frac{\sin\theta}{\theta} + L\sin 1''$ increases very slowly as θ increases, by taking for $\mathrm{l}_{10}\frac{\sin\theta}{\theta} + L\sin 1''$ the quantity which is given in the tables for the angle containing that number of seconds which is the nearest integer to the given number (n) and adding $\mathrm{l}_{10}n$ to it, the value of $L\sin n''$ is obtained without sensible error.

Thus; *If* n = 546·25, *required* L sin 546″·25.

By Taylor's Tables. $L\sin 546''$, or $L\sin 9', 6''$, is 7·4227670;

and $\mathrm{l}_{10}546 = 2{\cdot}7371926$.

Therefore, when θ is an angle of 546″,

$$\mathrm{l}_{10}\frac{\sin\theta}{\theta} + L\sin 1'' = L\sin n'' - \mathrm{l}_{10}n \quad \text{Art. 20: p. 127.}$$

$$= 7{\cdot}4227670 - 2{\cdot}7371926 = 4{\cdot}6855744,$$

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* For $\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{1.2.3} + \frac{\theta^4}{1.2.3.4.5} - \ldots\ = 1 - \frac{\theta^2}{6}$ nearly;

$\therefore\ \mathrm{l}_{10}\frac{\sin\theta}{\theta} = -\frac{1}{\mathrm{l}_\epsilon 10}\cdot\left\{\frac{\theta^2}{6} + \frac{1}{2}\left(\frac{\theta^2}{6}\right)^2 + \ldots\right\}$; $\ \therefore\ d_\theta \mathrm{l}_{10}\frac{\sin\theta}{\theta} = -\frac{1}{\mathrm{l}_\epsilon 10}\cdot\left(\frac{\theta}{3} + \frac{2\theta^3}{6^2} + \ldots\right) =$ a very small quantity, if 0 be an angle of a few minutes only.

the quantity corresponding to the angle 546″ which would be given in the tables of $l_{10} \frac{\sin\theta}{\theta} + L \sin 1''$.

$\therefore$ By (i.) Art. 21, p. 128, $L \sin 546''{\cdot}25 = 4{\cdot}6855744 + l_{10} 546{\cdot}25$

$$= 4{\cdot}6855744 + 2{\cdot}7373914 = 7{\cdot}4229658.$$

Ex. 2. *To determine* n *when* L sin n″ *is given.*

By referring to the tables of L sines in common use the integral number of seconds is found which is contained in the angle whose L sine is next below the proposed quantity: suppose this number to be m. Then substituting in (ii.) Art. 21, p. 128, the quantity given in the tables for that value of $\{l_{10} \frac{\sin\theta}{\theta} + L \sin 1''\}$ which corresponds to the angle m'', a near approximation to the value of n is obtained.

Thus, *If* L sin n″ = 7·4230612, *required* n.

By Taylor's Tables,

$$L \sin 546'' = 7{\cdot}4227670, \text{ and } L \sin 547'' = 7{\cdot}4235617;$$

$\therefore$ the value of m in this case is 546.

Now, as in the last example, the value of $l_{10} \frac{\sin\theta}{\theta} + L \sin 1''$ for an angle 546″ is 4·6855744.

Therefore by (ii.), Art. 21, p. 128,

$$l_{10} n = 7{\cdot}4230612 - 4{\cdot}6855744 = 2{\cdot}7374868 = l_{10} 546{\cdot}37;$$

and $\therefore$ 546″·37 is the angle sought.

EXAMPLES.

1. **Prove** that $45^{\circ}, 15', 20'' = 50^{g}, 28^{\backprime}, 39^{\backprime\backprime}\cdot 50$; $10^{\circ}, 15', 37'' = 11^{g}, 40^{\backprime}, 3^{\backprime\backprime}\cdot 09$; $18^{\circ}, 10', 48'' = 20^{g}, 20^{\backprime}$; $\dfrac{180^{\circ}}{\sqrt{3}} = 115^{g}, 47^{\backprime}$.

2. The Complements of $17^{\circ}, 36', 43''$; $29^{\circ}, 27', 6''\cdot 32$; and $216^{\circ}, 45'$; are $72^{\circ}, 23', 17''$; $60^{\circ}, 32', 53''\cdot 68$; and $-126^{\circ}, 45'$.

3. The Supplements of $37^{\circ}, 4', 3''$; $115^{\circ}, 13', 24''\cdot 66$; and $226^{\circ}, 14', 17''$; are $142^{\circ}, 55', 57''$; $64^{\circ}, 46', 35''\cdot 34$; and $-(46^{\circ}, 14', 17'')$.

4. If $\text{Cot}\, A = \dfrac{3}{2}$, find the values of $\text{Sin}\, A$, $\text{Cos}\, A$, $\text{Cosec}\, A$, $\text{Versin}\, A$, and $\text{Sec}\, A$.

5. Prove the formulæ,

(1) $\text{Sec}^2 A \,.\, \text{cosec}^2 A = \sec^2 A + \text{cosec}^2 A$.

(2) $\text{Cot}^2 A \,.\, \cos^2 A = \cot^2 A - \cos^2 A$.

(3) $\text{Cos}\, A = \dfrac{\cot A}{\sqrt{(1 + \cot^2 A)}}$. (4) $\text{Versin}\, A = \dfrac{\sec A - 1}{\sec A}$.

(5) $\text{Sin}\, A \,.\, \cos A = \dfrac{1}{\tan A + \cot A}$.

6. (1) If $\text{Tan}^2 A + 4 \sin^2 A = 6$; $A = 60^{\circ}$.

(2) If $m = \tan A + \sin A$, and $n = \tan A - \sin A$; $\text{Cos}\, A = \dfrac{m-n}{m+n}$.

(3) If $m \,.\, \sin A = n \,.\, \cos A$; $\text{Sin}\, A = \pm \dfrac{n}{\sqrt{m^2 + n^2}}$.

(4) If $1 = \left(\dfrac{\sin A}{\sin B}\right)^2 + (\cos A \,.\, \cos C)^2$; $\text{Sin}\, C = \dfrac{\tan A}{\tan B}$.

(5) If $\text{Cos}\, x = \dfrac{\cos A}{\sin C}$, and $\text{Cos}\,(90^{\circ} - x) = \dfrac{\cos B}{\sin C}$;

$$\text{Cos}^2 A + \cos^2 B + \cos^2 C = 1.$$

7. Prove the following formulæ:

(1) $\text{Tan}\,A + \cot A = 2\,\text{cosec}\,2\,A$. (2) $\text{Cot}\,A - \tan A = 2\cot 2\,A$.

(3) $\text{Tan}\,A = \dfrac{\sin 2\,A}{1 + \cos 2\,A}$. (4) $\sqrt{\dfrac{1 + \sin A}{1 - \sin A}} = \dfrac{1 + \tan\frac{1}{2}A}{1 - \tan\frac{1}{2}A}$.

(5) $\text{Cosec}\,2\,A + \cot 2\,A = \cot A$. (6) $2\,.\,\text{Cosec}\,2\,A = \sec A\,.\,\text{cosec}\,A$.

(7) $\dfrac{\text{Sin}\,A}{1 - \cos A} = \cot\frac{1}{2}A$. (8) $\dfrac{\text{Versin}\,A}{\text{versin}\,(180^\circ - A)} = \tan^2\frac{1}{2}A$.

(9) $8\,.\,\cot 2\,A\,.\,\text{cosec}^2\,2\,A = \cot A\,.\,\text{cosec}^2 A - \tan A\,.\,\sec^2 A$.

(10) $\text{Versin}\,(180^\circ - A) = 2\,\text{vers}\,\frac{1}{2}(180^\circ + A)\,.\,\text{vers}\,\frac{1}{2}(180^\circ - A)$.

(11) $\text{Sec}\,2\,A = \dfrac{\cot A + \tan A}{\cot A - \tan A}$. (12) $\text{Tan}^2\frac{1}{2}A = \dfrac{2\sin A - \sin 2\,A}{2\sin A + \sin 2\,A}$.

8. Prove that, whatever be the values of the angles,

(1) $\text{Cos}\,2\,A + \cos 2\,B = 2\,.\,\cos(A + B)\,.\,\cos(A - B)$.

(2) $1 + \cos 2\,A\,.\,\cos 2\,B = 2\,.\,(\sin^2 A\,.\,\sin^2 B + \cos^2 A\,.\,\cos^2 B)$.

(3) $\text{Cos}^2(A + B) - \sin^2 A = \cos B\,.\,\cos(2\,A + B)$.

(4) $\dfrac{\text{Sin}\,(A - B)}{\sin A\,.\,\sin B} + \dfrac{\sin(B - C)}{\sin B\,.\,\sin C} + \dfrac{\sin(C - A)}{\sin C\,.\,\sin A} = 0$.

(5) $\text{Cos}\,(A + B)\,.\,\sin(A - B) + \cos(B + C)\,.\,\sin(B - C)$

$+ \cos(C + D)\,.\,\sin(C - D) + \cos(D + A)\,.\,\sin(D - A) = 0$.

(6) $\text{Cos}\,(A + B)\,.\,\sin B - \cos(A + C)\,.\,\sin C$

$= \sin(A + B)\,.\,\cos B - \sin(A + C)\,.\,\cos C$.

(7) $\text{Sin}\,(A + B)\,.\,\cos B - \sin(A + C)\,.\,\cos C = \sin(B - C)\,.\,\cos(A + B + C)$.

(8) $\text{Sin}\,(A + B - 2\,C)\,.\,\cos B - \sin(A + C - 2\,B)\,.\,\cos C$

$= \sin(B - C)\,.\,\{\cos(B + C - A) + \cos(A + C - B) + \cos(A + B - C)\}$.

9. Prove that

(1) $\dfrac{1+\sin A}{1-\sin A} = \tan^2(45^\circ + \frac{1}{2}A)$.

(2) $\text{Sec}\,(45^\circ + A)\,.\,\sec(45^\circ - A) = 2\sec 2A$.

(3) $2\sec A = \tan(45^\circ + \frac{1}{2}A) + \cot(45^\circ + \frac{1}{2}A)$.

(4) $\text{Tan}\,(30^\circ + A)\,.\,\tan(30^\circ - A) = \dfrac{2\cos 2A - 1}{2\cos 2A + 1}$.

(5) $\text{Sin}\,(60^\circ + A) - \sin(60^\circ - A) = \sin A$.

10. Find the numerical values of Sin 15°, Sin 9°, Cos 12°, Versin 15°, Tan 22°, 12′, Sin 150°, Cos 135°, Sin 3°, Sin 75°, Sec 225°; and prove that

(1) $\text{Tan}\,50^\circ + \cot 50^\circ = 2\sec 10^\circ$.

(2) $2\cos\dfrac{A}{2^n} = \sqrt{2 + \sqrt{\{2 + \&c + \sqrt{(2 + 2\cos A)}\}}}$; where n is an integer, and the symbol $\sqrt{}$ is repeated n times, each time affecting all the quantities which follow it.

11. Determine A in the following equations:

(1) $\text{Sin}\,A = \sin 2A$.

(2) $\text{Tan}\,2A = 3\tan A$.

(3) $\text{Tan}\,A = \text{cosec}\,2A$.

(4) $\text{Cos}\,A = \tan A$.

(5) $\text{Tan}^2 2A = 3\tan 2A$.

(6) $\text{Tan}\,A + 3\cot A = 4$.

(7) $\text{Tan}\,A\,.\,\tan 2A + \cot A = 2$.

(8) $2\sin^2 3A + \sin^2 6A = 2$.

(9) $\text{Tan}\,A + \cot 2A = \sin A\,.\,(1 + \tan A\,.\,\tan\frac{1}{2}A)$.

(10) $\text{Cos}\,nA + \cos(n-2)A = \cos A$.

12. Determine x in the following equations:

(1) $\text{Sin}\,(a - x) = \cos(a + x)$.

(2) $\text{Sin}\,(x + a) + \cos(x + a) = \sin(x - a) + \cos(x - a)$.

(3) $\text{Sin}\,\alpha + \sin(x-\alpha) + \sin(2x+\alpha) = \sin(x+\alpha) + \sin(2x-\alpha)$.

(4) $\text{Tan}\,\alpha \,.\, \tan x = \tan^2(\alpha + x) - \tan^2(\alpha - x)$;—to find $\text{Cos}\,x$.

(5) $m \,.\, \text{versin}\, x = n \,.\, \text{versin}\,(\alpha - x)$.

(6) $\dfrac{m \,.\, \tan(\alpha - x)}{\cos^2 x} = \dfrac{n \,.\, \tan x}{\cos^2(\alpha - x)}$. [*Ans.* $\text{Tan}\,(\alpha - 2x) = \dfrac{n-m}{n+m} \,.\, \tan\alpha$.]

(7) $\text{Tan}\,(\alpha + x) \,.\, \tan(\alpha - x) = \dfrac{1 - 2\cos 2\alpha}{1 + 2\cos 2\alpha}$. [$x = 30^0$.]

(8) $n \,.\, \sec^2 x \,.\, \tan(\alpha - x) = m \,.\, \sec^2(\alpha - x) \,.\, \tan x$.

$$\left[\text{Tan}\,2x = \frac{m \sin 2\alpha}{m + n\cos 2\alpha}.\right]$$

(9) $\text{Sin}\,x = \sin\alpha \,.\, \sin(\beta + x)$. $\left[\text{Tan}\,x = \dfrac{\sin\alpha \,.\, \sin\beta}{1 - \sin\alpha \,.\, \cos\beta}.\right]$

(10) $\text{Tan}\,x = \cos\alpha \,.\, \tan\beta$. Shew that

$$\text{Tan}\,(\beta - x) = \frac{\tan^2 \frac{1}{2}\alpha \,.\, \sin^2 2\beta}{1 + \tan^2 \frac{1}{2}\alpha \,.\, \cos 2\beta}.$$

(11) $\dfrac{m}{n} = \dfrac{\sin\alpha}{\sin\beta} \,.\, \dfrac{\cos(\beta + x)}{\cos(\alpha - x)}$. Find $\text{Tan}\,x$.

(12) If $2\sin(x - y) = 1$, and $\text{Sin}\,(x - y) = \cos(x + y)$;

then $x = 45^0$, and $y = 15^0$.

13. If $A + B + C = 90^0$;

(1) $\text{Tan}\,A \,.\, \tan B + \tan A \,.\, \tan C + \tan B \,.\, \tan C = 1$.

(2) $\text{Cot}\,A + \cot B + \cot C = \cot A \,.\, \cot B \,.\, \cot C$.

(3) $\text{Tan}\,A + \tan B + \tan C = \tan A \,.\, \tan B \,.\, \tan C + \sec A \,.\, \sec B \,.\, \sec C$.

14. If $A + B + C = 180^0$;

(1) $4\sin A \,.\, \sin B \,.\, \sin C = \sin 2A + \sin 2B + \sin 2C$.

(2) $\text{Sin}^2 \frac{1}{2} A + \sin^2 \frac{1}{2} B + \sin^2 \frac{1}{2} C + 2\sin\frac{1}{2} A \,.\, \sin\frac{1}{2} B \,.\, \sin\frac{1}{2} C = 1$.

15. Prove that,

(1) $\text{Tan}^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 45^\circ$. (2) $\text{Tan}^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3} = 45^\circ$.

(3) $\text{Cot}^{-1}\frac{3}{4} + \cot^{-1}\frac{1}{7} = 135^\circ$. (4) $\text{Sin}^{-1}\frac{1}{\sqrt{5}} + \cot^{-1} 3 = 45^\circ$.

(5) If $\theta = \tan^{-1}\frac{1}{\sqrt{3}}$, and $\phi = \tan^{-1}\frac{1}{\sqrt{15}}$;

then $\text{Sin}(\phi + \theta) = \sin 60^\circ . \cos 36^\circ$.

(6) If $\theta = \cos^{-1}\sqrt{\frac{2}{3}}$, and $\phi = \cos^{-1}\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{2}}$; then $\phi + \theta = 60^\circ$.

(7) If $\text{Cot}\, x = n . \cot(\alpha - x)$; then $2x = \alpha - \sin^{-1}\left(\frac{n-1}{n+1} . \sin\alpha\right)$.

(8) If $\text{Versin}^{-1}(1+x) - \text{versin}^{-1}(1-x) = \tan^{-1} 2\sqrt{1-x^2}$;

then $x = \pm 1$, or $-\frac{1}{2}$.

(9) If $\text{Versin}^{-1}\frac{x}{a} - \text{versin}^{-1}\frac{bx}{a} = \text{versin}^{-1}(1-b)$; then $\frac{x}{a} = \pm\sqrt{\frac{2b}{1+b}}$.

16. (1) If $m = \tan\theta - \sin\theta$, and $n = \tan\theta + \sin\theta$; find the relation between m and n.

(2) If $\text{Tan}\,\theta = \frac{b}{a}$, then $a . \cos 2\theta + b . \sin 2\theta = a$.

(3) If $(x-a) . \cos\delta + c . \sin(l - \delta) = 0$,

$(y-b) . \cos\delta + c . \cos(l - \delta) = 0$;

then $(x-a) . \sin l + (y-b) . \cos l + c = 0$.

(4) If $a . \sin^2\theta + a' . \cos^2\theta = b$, $a' . \sin^2\theta' + a . \cos^2\theta' = b'$;

and $a . \tan\theta = a' . \tan\theta'$; then $\frac{1}{a} + \frac{1}{a'} = \frac{1}{b} + \frac{1}{b'}$.

(5) If $\text{Tan}\,\theta = \tan^3 \tfrac{1}{2}\phi$, and $\text{Cos}^2\phi = \dfrac{m^2 - 1}{3}$;

$$\text{then} \quad m = \frac{2}{(\cos^{\frac{2}{3}}\theta + \sin^{\frac{2}{3}}\theta)^{\frac{3}{2}}}.$$

(6) If $\text{Cos}\,v = \dfrac{\cos u - e}{1 - e\cos u}$; then $\text{Tan}\,\tfrac{1}{2}v = \sqrt{\dfrac{1+e}{1-e}}\,.\tan\tfrac{1}{2}u.$

17. In a RIGHT-ANGLED TRIANGLE ABC, C being the right angle;

(1) $\text{Cos}\,(2A - B) = \dfrac{a}{c^3}\,.\,(3c^2 - 4a^2).$

(2) Area = (semi-perimeter) × (semi-perimeter – hypothenuse).

(3) The perimeter being given, and the perpendicular from C on c, to find the three sides.

(4) A person observes the elevation of a tower to be 60°, and on receding from it 100 yards further he finds the elevation to be 30°; required the height of the tower.

(5) A cylindrical tower of given diameter terminates in a conical spire. At a measured distance from the foot of the tower the angles of elevation of the tops of the spire and the tower are observed. Find the heights of the tower and the spire, (1) when the measured distance is horizontal, and (2) when it is inclined at a given angle to the horizon.

(6) To a person standing 288 yards from the foot of a tower, its elevation is 53°, 8′; determine the logarithm of the height of the tower:

$$L\tan 53^\circ,\, 8' = 10{\cdot}1249818\,; \quad l_{10}288 - 2{\cdot}4593925.$$

(7) The hypothenuse is 7854·3, and one angle is 37°, 18′; find the logarithm of the opposite side.

$$L\,\text{cosec}\,37^\circ,\, 18' = 10{\cdot}2175357\,; \quad l_{10}7{\cdot}8543 = {\cdot}8951075.$$

18. General Properties.

(1) $\mathrm{Tan}\,B = \dfrac{b\,.\,\sin C}{a - b\,.\,\cos C}$.

(2) $\mathrm{Cos}\,\frac{1}{2}(A-B) = \dfrac{a+b}{c}\,.\,\sin\frac{1}{2}C$; $\mathrm{Sin}\,(A-B) = \dfrac{a^2-b^2}{c^2}\,.\,\sin C$.

(3) $\frac{1}{2}(a^2+b^2+c^2) = ab\,.\,\cos C + ac\,.\,\cos B + bc\,.\,\cos A$.

(4) $\frac{1}{2}(a+b+c) = \frac{1}{2}(a+b)\,.\,\cos C + \frac{1}{2}(a+c)\,.\,\cos B + \frac{1}{2}(b+c)\,.\,\cos A$.

(5) $c^2 = (a+b)^2\,.\,\sin^2\frac{1}{2}C + (a-b)^2\,.\,\cos^2\frac{1}{2}C$.

(6) $2a\,.\,\cot\frac{1}{2}A = (b+c-a)\,.\,(\cot\frac{1}{2}B + \cot\frac{1}{2}C)$.

(7) $\dfrac{a}{b}\,.\,\sin^2\frac{1}{2}B = \dfrac{S-a}{S-b}\,.\,\sin^2\frac{1}{2}A$.

(8) The length of the line from C which bisects c is

$$\tfrac{1}{2}\,.\,\{2(a^2+b^2) - c^2\}^{\frac{1}{2}}.$$

(9) $\mathrm{Area} = \frac{1}{2}(a^2-b^2)\,.\,\dfrac{\sin A\,.\,\sin B}{\sin(A-B)}$.

(10)
$$\mathrm{l}_e\frac{a}{b} = (\cos 2A - \cos 2B) + \tfrac{1}{2}\,.\,(\cos 4A - \cos 4B)$$
$$+ \tfrac{1}{3}\,.\,(\cos 6A - \cos 6B) + \dots$$

(11) $\mathrm{l}_e\dfrac{a}{c} = \dfrac{b}{a}\,.\,\cos C + \dfrac{b^2}{2a^2}\,.\,\cos 2C + \dfrac{b^3}{3a^3}\,.\,\cos 3C + \dots$

(12) $B'' = \dfrac{b}{a}\,.\,\dfrac{\sin C}{\sin 1''} + \dfrac{b^2}{a^2}\,.\,\dfrac{\sin 2C}{\sin 2''} + \dfrac{b^3}{a^3}\,.\,\dfrac{\sin 3C}{\sin 3''} + \dots$

19. (1) The sides of a triangle are 13, 12, 5; find its area.

(2) Given the perimeter, area, and an angle, find the side opposite to the given angle.

(3) Given B, a, and the area (S), to determine the triangle.

(4) Given the vertical angle, the perpendicular on the base, and the rectangle under the segments of the base, to determine the triangle.

(5) The length of a road, in which the ascent is 1 foot in five, from the foot of a hill to the top is a mile and two-thirds. What will be the length of a zigzag road in which the ascent is 1 foot in 12?

(6) The area of a triangle is $\frac{4}{3}$ of that triangle whose sides are equal to the three lines which join the angles of the first triangle with the bisections of the sides respectively opposite.

(7) If in the three edges of a cube which meet in a point O, points A, B, C be taken which are distant a, b, c from O, the area of the triangle formed by joining A, B, C is $\frac{1}{2}(a^2b^2+b^2c^2+a^2c^2)^{\frac{1}{2}}$.

(8) The sides of a triangle are as $n-1$, n, $n+1$, where n is very large; find the angles, and determine the difference of each from 60^0.

(9) An object six feet high, standing at the top of a tower, subtends an angle $\text{Tan}^{-1}\cdot 015$ at a station which is 100 yards from the base. Find the height of the tower. *Ans.* $\pm\frac{1}{3}\sqrt{30009}-1$. Explain the meaning of the double result.

(10) Two objects, A and B, were observed to be at the same instant in a line inclined at an angle 15^0 to the east of a ship's course which was at the time due north. The ship's course was then altered, and after sailing 5 miles in a N. W. direction, the same objects were observed to bear E. and N. E. respectively. Required the distance of A from B. *Ans.* $5\,.\,(3-\sqrt{3})$ miles.

(11) A vessel observed another α^0 from the north, sailing in a direction parallel to its own. In p hours its bearing was β^0, and in q hours afterwards γ^0 from the north. To what point of the compass were the vessels sailing?

(12) The shadows of two vertical walls, which are at right angles to each other, and are a and a_1 feet in height, are observed, when the Sun is due south, to be b and b_1 feet in breadth. Shew that if α be the Sun's altitude above the horizon, and β be the inclination of the first wall to the meridian;

$$\text{Cot}\,\alpha=\sqrt{\left(\frac{b^2}{a^2}+\frac{b_1^{\,2}}{a_1^{\,2}}\right)}. \qquad \text{Cot}\,\beta=\frac{a\,b_1}{a\,b}.$$

(13) A boy was flying a kite at noon when the wind was blowing α^0 from the south, and the angular distance of the kite's shadow from the north was β^0. The wind suddenly changed to $\alpha_{\prime}^{\,0}$ from the south, and the shadow to $\beta_{\prime}^{\,0}$ from the north,

and the kite was raised as much above 45° as it had before been below that elevation. Shew that θ^0 being the angular elevation of the Sun, and $45^0 - \phi^0$ that of the kite at first,

$$\text{Tan}^2\,\theta^0 = \frac{\sin\beta\,.\,\sin\beta_{,}}{\sin(\alpha-\beta)\,.\,\sin(\alpha_{,}-\beta_{,})},$$

$$\text{Tan}^2\,(45^0-\phi^0) = \frac{\sin(\alpha-\beta)\,.\,\sin\beta_{,}}{\sin(\alpha_{,}-\beta_{,})\,.\,\sin\beta}.$$

(14) A privateer observes the direction in which a trader leaves a harbour at a known distance. Having given the rates of sailing of the two vessels, shew how to compute the angle at which the privateer must pursue the trader, so that she may just have prepared for action at the time she comes within gun-shot.

(15) There are two towns lying N. and S. of each other, and distant $2\frac{1}{4}$ miles. Their angles of depression as observed from a balloon are 45° and 60°, and after the balloon has proceeded horizontally 6 miles in a S. E. direction, the angles of depression of each are $\frac{1}{2}$ of what they were before. Shew that the height of the balloon was 3 miles very nearly.

(16) The sides a, b, c of a triangle are 4, 5, 6, find the angle B; having given

$l_{10}2 = \cdot 3010300.$ $L\cos 27^\circ, 53' = 9\cdot 9464040.$

$l_{10}5 = \cdot 6989700.$ $L\cos 27^\circ, 54' = 9\cdot 9463371.$

(17) Given $l_{10}3 = \cdot 4771213$, and $L\tan 57^\circ, 19', 11'' = 10\cdot 1928032$, nearly. Shew that if one angle be 60°, and the two sides containing it are as 19 to 1, the other two angles are 117°, 19′, 11″, and 2°, 40′, 49″.

(18) Two sides of a triangle are as 9 to 7, and the included angle is 64°, 12′; determine the degrees, minutes, and seconds in the other angles. Given

$l_{10}2 = \cdot 3010300,$ $L\tan 57^\circ, 54' = 10\cdot 2025255.$

$L\tan 11^\circ, 16' = 9\cdot 2993216.$ $L\tan 11^\circ, 17' = 9\cdot 2999804.$

20. (1) The radii of the circles described within and about a triangle whose sides are a, b, c being r and R, prove that

$$2Rr = \frac{abc}{a+b+c}; \quad \text{and} \quad r = 4R\,.\,\sin\tfrac{1}{2}A\,.\,\sin\tfrac{1}{2}B\,.\,\sin\tfrac{1}{2}C.$$

(2) The product of the radii of the four circles which touch three lines intersecting each other = the square of the area of the triangle which the three lines include.

(3) Three circles whose radii are a, b, c, touch each other externally; prove that the tangents at the points of contact meet in a point, whose distance from any one of the points of contact is

$$\left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}.$$

(4) The areas of all triangles described about a given circle are as their perimeters.

(5) If a, a' be homologous sides of similar triangles described about and within a circle, $a' = 4a \,.\, \sin \tfrac{1}{2} A \,.\, \sin \tfrac{1}{2} B \,.\, \sin \tfrac{1}{2} C$.

21. (1) In a regular polygon of n sides, of which a side is $2a$, $R + r = a \,.\, \cot \dfrac{\pi}{2n}$.

(2) The square of a side of a pentagon inscribed in a circle = the square of a side of the inscribed hexagon, together with the square of the side of the inscribed decagon.

(3) The area of a circle and that of a regular polygon inscribed in it being given, find the number of the sides of the polygon, and the angles they make one with another.

22. Adapt to logarithmic computation;

(1) $\sqrt{a+b} + \sqrt{a-b}$. (2) $\dfrac{\sqrt{a+b}}{a-b} + \dfrac{\sqrt{a-b}}{a+b}$.

In the following questions θ is required.

(3) $\operatorname{Sin}(\theta + a) = m \,.\, \sin \theta$.

(4) $\operatorname{Cos}(a + \theta) = \cos a \,.\, \sin \theta + \sin \beta$. (5) $\operatorname{Cos} m\theta + \cos n\theta + \cos p\theta = 1$.

(6) $\operatorname{Tan} \theta = \dfrac{na \,.\, \cos A - b \,.\, \sin B}{ma \,.\, \cos A + c \,.\, \sin C}$; where A, B, C are the angles of a triangle, and a, b, c the sides.

(7) $\operatorname{Sin} \theta \,.\, \sqrt{(1 + \tan^2 A \,.\, \tan^2 B)} + \cos \theta \,.\, \sqrt{(1 - \tan^2 A \,.\, \tan^2 B)}$
$= \tan A + \tan B$.

23. Prove that,

(1) $\tfrac{2}{3} \,.\, (8 \sin \tfrac{1}{4}\theta - \sin \tfrac{1}{2}\theta) = \theta$; nearly.

(2) $\operatorname{Versin} \theta = -\tfrac{1}{2} \,.\, \{\epsilon^{\frac{1}{2}\theta\sqrt{-1}} - \epsilon^{-\frac{1}{2}\theta\sqrt{-1}}\}$.

(3) If from an arc AB of a circle of radius r whose centre is O, AC be cut off $= r \,.\, \sin AOB$; then sector $COB =$ segment ACB.

(4) The tops of two vertical rods on the Earth's surface, each of which is 10 feet high, cease to be visible from each other when 8 miles distant. Prove that the Earth's radius is nearly 4224 miles.

(5) The top of a mountain may be just seen from a ship at sea which is a miles off, and at a distance of b miles the angular elevation of the mountain above the horizon is θ. Prove from these data that the Earth's radius is $\frac{a^2 - b^2}{2b} \,.\, \cot \theta$ nearly, and determine the height of the mountain.

24. Solve the equations;

(1) $x^6 + 1 = 0$.

(2) $x^3 - 6x - 4 = 0$. [$x = -2$, or $1 \pm \sqrt{3}$.]

(3) $x^3 - 147x - 343 = 0$. [One root is 13·1556964; given that Cos 20° = ·9396926.]

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25. LOGARITHMS.

(1) From $l_{10}2$, and $l_{10}3$, find $l_{10}72$.

(2) From $l_{10}14$, and $l_{10}16$, find $l_{10}2$, $l_{10}5$, $l_{10}7$, $l_{10}\frac{1}{1960}$.

(3) From $l_{10}2$, and $l_{10}67$, find $L \tan 15^\circ$.

(4) $l_\epsilon\{(1 + N)^{\frac{1+N}{2}} \,.\, (1 - N)^{\frac{1-N}{2}}\} = \frac{N^2}{1 \,.\, 2} + \frac{N^4}{3 \,.\, 4} + \frac{N^6}{5 \,.\, 6} + \dots$

THE ELEMENTS

OF

SPHERICAL TRIGONOMETRY.

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SPHERICAL TRIGONOMETRY.

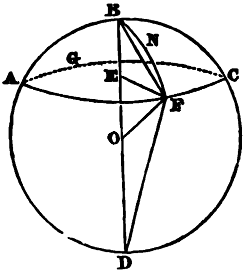
CHAPTER I.

ON CERTAIN PROPERTIES OF SPHERICAL TRIANGLES.

1. Def. A *Sphere* is a solid bounded by a surface of which every point is equally distant from a fixed point which is called the *centre* of the sphere.

Def. The line joining the centre with any point in the surface is called the *radius* of the sphere.

2. *Every section of a sphere made by a plane is a circle.*



Let $ABCD$ be a sphere of which the centre is O; $AFCG$ the curve in which a plane cutting the sphere intersects its surface; OE a perpendicular from O upon the cutting plane.

Join E with F any point in $AFCG$, and join FO. Then since OE is perpendicular to the cutting plane, it is perpendicular to EF, a line meeting it in that plane;

$\therefore EF = \sqrt{(OF^2 - OE^2)}$, a constant quantity.

Now E is a *fixed* point in the cutting plane, and F is *any* point in the curve AFC. Therefore AFC is a circle whose centre is E and radius is EF. Euclid, I. Def. 15.

Cor. If the cutting plane pass through the centre of the sphere, EO vanishes, and EF becomes equal to OF, the radius of the sphere.

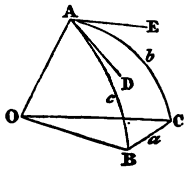
3. Def. The circle in which a sphere is cut by a plane is called a *great*, or a *small*, circle according as the cutting plane passes, or does not pass, through the centre of the sphere.

Cor. Since the radius of every great circle is equal to the radius of the sphere (Art. 2, Cor.), all great circles of the sphere must be equal.

Note. Unless the contrary be expressly mentioned, when hereafter an arc of a sphere is spoken of, an arc of a *great* circle is meant.

4. Def. Spherical Trigonometry investigates the relations subsisting between the angles of the plane faces which form a solid angle, and the angles at which the plane faces themselves are inclined to one another.

For the sake of convenience the angular point is made the centre of a sphere, which being cut by the plane faces containing the solid angle, presents on its surface a figure of which the sides are arcs of great circles.



Let *ABC* be a triangle of this kind, whose sides *AB, BC, CA*, are formed by the intersection of the planes *AOB, BOC, COA*, with the surface of a sphere of which *O* is the centre. The angle of any face, as *AOB*, is $\frac{\text{arc}\,AB}{AO}$, Pl. Trig. Chap. vi.; and the angle contained between any two faces, (as *BOA* and *COA*), is the angle contained between *AD* and *AE*, which are lines drawn in the planes *BOA* and *COA* at right angles to their intersection *OA*. Euclid xi. Def. 6.

The lines *AD, AE*, being at right angles to the radius *OA*, and lying in the planes of the arcs *AB* and *AC*, are tangents to those arcs.

5. After certain properties of spherical triangles have been proved, it will not be requisite, in pursuing further investigations, to represent in the figures attached to the Propositions the centre of the sphere on which the triangles are described.

It must not, however, be forgotten, that when the words "the angle *BAC*" or "the angle *A*" occur, the angle meant is that of the inclination of the planes passing through *O* the centre of the sphere and the arcs *AB* and *AC*; and that this is *the angle contained between two lines drawn from any point in* AO *at right angles to it, and respectively lying in the planes* AOB *and* AOC. Also, whenever the expression "the sine of *AB*" occurs, the sine is meant of the angle which the arc *AB* subtends at the centre of the circle of which it is a portion. If this circle be a great circle, its centre is also the centre of the sphere.

6. In the following pages the angle *BAC* will commonly be indicated by the letter *A*, and the angle subtended by *BC*, the side of the triangle opposite to ∠*BAC*, will be indicated by *a*. The other parts of the triangle *ACB* will be represented in like manner.

7. Def. If OE (fig. Art. 2.), which is perpendicular to the plane AFC, be produced both ways to meet the surface of the sphere in B and D, these points are respectively called *the nearer* and *the further poles* of the circle AFC; and the straight line BOD is called *the axis of the circle AFC.*

8. *The pole of a circle is equally distant from every point in its circumference.* (Fig. Art. 2.)

Join B with F any point in $AFCG$. Then, BEF being a right angle,

$$BF^2 = BE^2 + EF^2$$

$$= BE^2 + (OF^2 - OE^2); \text{ a constant quantity.}$$

And F being *any* point in $AFCG$, B is equally distant from every point in the circumference of that circle.

Similarly, $DF^2 = DE^2 + EF^2$

$$= DE^2 + (OF^2 - OE^2); \text{ a constant quantity.}$$

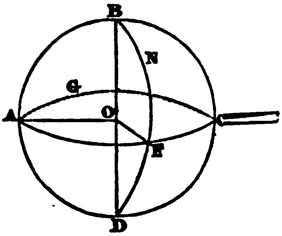
Hence D is equally distant from every point in the circumference of $AFCG$.

Again; let BF be an arc of a great circle passing through B and F. Then since that BF is constant, and that in equal circles equal circumferences are subtended by equal straight lines (Euclid III. xxix.), the arc BNF is constant; and also, because the radii of all great circles are equal, the angle BOF subtended at the centre of the sphere is constant.

Hence it appears that every point in the circumference of a circle of a sphere is equally distant from the pole of the circle, whether the distance be measured by the length of a straight line joining the point with the pole, or by the arc of a great circle connecting the same points, or by the angle which this arc subtends at the centre of the sphere.

9. Since BO is at right angles to the plane AFC, every plane through BO is at right angles to that plane. Hence, *the angle between any circle whatever and a great circle passing through its pole is a right angle.*

10. *If* FB *be a quadrant drawn from any point* F *in th great circle* AFC *at right angles to it,* B *is the pole of* AFC.



Join *B* and *F* with *O* the centre of the sphere, and in the plane of *AFC* draw *OA* at right angles to *FO*.

Then, *AO* being perpendicular to *OF*, and (since *FB* is a quadrant) *BO* being perpendicular to the same line, therefore ∠ *BOA* = ∠ between the planes *AOF* and *BOF*, whose intersection is *OF*; an angle which is by hypothesis a right angle.

And *BO*, being at right angles to *OA* and *OF*, is at right angle to the plane *AOF* in which they lie; and therefore *B* is the pole o *AFC*.

Cor. If *B* be the pole of *AFC*, *AO* and *FO* are at right angles to *BO*, and *AOF* is the inclination of *AOB* and *FOB*;

$$\therefore \text{ Art. } 5, \quad \angle AF = \angle ABF.$$

Def. Great circles which pass through the pole of a great circle are called *secondaries* to that circle.

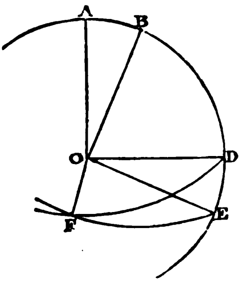
11. *If from a point on the surface of a sphere there can be drawn two arcs* (*not parts of the same circle*) *which are at righ angles to a given circle, that point is the pole of the circle.* (Fig Art. 10.)

Let *B* be the point, *BA* and *BF* arcs through *A* and *F*, points i the circle *AFC*; and let *BA* and *BF* be at right angles to *AF*.

Then since *BA* and *BF* are at right angles to *AF*, and are no parts of the same circle, the planes *BOA* and *BOF* must intersect and their intersection *BO* is at right angles to *AOF*, Eucl. xi. 19 Therefore *B* is the pole of *AFC*.

Cor. Hence also it appears that the intersection of two arcs drawn at right angles to a given circle through any two points i it, is the pole of that circle.

12. *The angle subtended at the centre of the sphere by th arc which joins the poles of two great circles, is the inclinatio of the planes of the circles.*



Let the given circles be FD and FE intersecting in F, A and B their respective poles, and $ABDE$ the circle through A and B.

Now AO is perpendicular to OF, which is a line in the plane DOF,

And BO is perpendicular to OF, a line in the plane EOF;

$\therefore$ OF is perpendicular to the plane AOB; and therefore to OD and OE, which are lines in that plane;

$\therefore$ DOE is the angle of inclination of the planes FOD, FOE.

$$\begin{aligned} \text{And } AOB &= AOD - BOD = 90^\circ - BOD \\ &= BOE - BOD \\ &= DOE. \end{aligned}$$

Cor. Hence also it appears that $\angle AB = \angle DE = \angle DFE$. Arts. 5, 4.

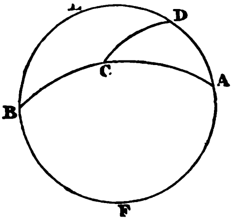
13. *Two great circles are bisected by their intersections.* (Fig. Art. 10.)

Let BAD and BFD be arcs of two great circles intersecting each other in B and D. Join BD.

Then since BAD is a great circle, the centre of the sphere is a point in its plane; similarly, it is a point in the plane of BFD; therefore the centre of the sphere is a point O which is in the intersection of these planes, that is, in the line joining B and D. Therefore BAD and BFD are semicircles, having BOD for their common diameter, and the two great circles are bisected by the points B and D.

14. *Any side of a spherical triangle is less than a semicircle, and any angle is less than two right angles.*

Since Euclid takes two right angles as the limit of a plane angle, this is also his limit for the angle of any plane face of a solid angle. Hence in a spherical triangle no side can be equal to a semicircle.



Thus, if ACB intersect $AEBF$ in the points A and B, and CD be any other arc, the triangle connecting the points B, C, D is *not* the figure formed by BC, CD, $DAFB$ (of which the side $DAFB$ is greater than a semicircle), but the figure formed by BC, CD, DEB.

If a side, as *ADEB*, become a semicircle, then the arcs which join *A* and *B* with any other point *C*, are portions of the semicircle *BCA*, and the arcs joining the points *A*, *B*, *C* cease to form a triangle.

Wherefore the *side* of a triangle is less than a semicircle.

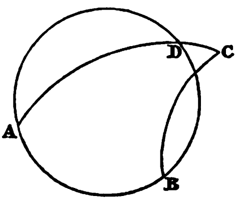
Cor. 1. Hence it follows that an *angle* of a triangle must be less than two right angles.

For if possible let *BFADC* be a triangle having the angle *BCD* greater than two right angles. Produce *BC* to *A*; then *BFA* is a semicircle, and *BFAD* is greater than a semicircle; which is impossible.

If the angle *BCD* become equal to two right angles, *BC* and *CD* become parts of the same circle, and the figure ceases to be a triangle.

Wherefore the *angle* of a triangle is less than two right angles.

Cor. 2. Hence also it follows, that if the great circle be completed of which any side *AB* of a triangle *ABC* forms a part, the triangle lies within it.



For if possible, let *C* lie without the circle, and let *CA* cut it in *D*; then since *DA* is a semicircle (13), *CDA* is greater than a semicircle; which is impossible.

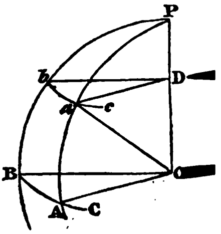
15. *If* P *be the pole of a great circle* BAC *and of a small circle* bac *which are cut by the great circles* PaA *and* PbB, *then*

$$\frac{\text{Arc ab}}{\text{Arc AB}} = \text{Sin Pa}.$$

Let *OP*, which is at right angles to the planes of both circles, cut the plane of the small circle in *D*. Join *AO*, *BO*, *aD*, *bD*. Since these lines lie in planes which are perpendicular to *OP*, each of them is perpendicular to *OP*. Join *aO*.

Then *aD*, being perpendicular to *PO*, is parallel to *AO* which lies in the same plane with it.

Similarly, *bD* is parallel to *BO*;



$$\therefore \angle aDb = \angle AOB. \quad \text{Eucl. xi. 10,}$$

$$\text{that is, } \frac{\text{arc}\, ab}{aD} = \frac{\text{arc}\, AB}{AO};$$

$$\therefore \frac{\text{Arc}\, ab}{\text{Arc}\, AB} = \frac{aD}{AO} = \frac{aD}{a\,O} = \sin POa, \quad \text{or } \sin Pa.$$

Cor. 1. Since $POA = 90^\circ$, $\sin POa = \cos aOA$;

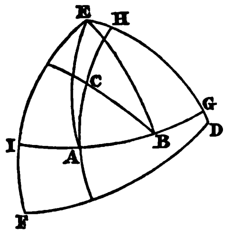
$$\therefore \frac{\text{Arc } ab}{\text{Arc } AB} = \cos aOA, \quad \text{or} \quad \cos aA.$$

Cor. 2. If PA be not a quadrant, it may be proved in like manner that

$$\frac{\text{Arc } ab}{\text{Arc } AB} = \frac{\sin Pa}{\sin PA}.$$

16. **Def.** If the angular points A, B, C of a spherical triangle ABC, Fig. Art. 17, be the poles of the three great circles DE, EF, FD respectively, the triangle ABC is called with respect to DEF the *Primitive* Triangle, and DEF is called with respect to ABC the *Polar* Triangle.

17. *The angular points of the polar triangle are the poles of the sides of the primitive triangle.*



Join AE and BE, by arcs of two great circles.

Then, since A is the pole of ED, $AE = \frac{1}{2}\pi$, Art. 10.

And, since B is the pole of EF, $BE = \frac{1}{2}\pi$.

Therefore the great circle of which E is the pole passes through A and B, or E is the pole of AB. Similarly D and F are the poles of AC and BC respectively.

18. The *Polar* Triangle is called also the *Supplemental* Triangle, from the following property it possesses.

The sides and angles of the polar triangle are the supplements of the angles and sides respectively of the primitive triangle. (Fig. Art. 17.)

$$\text{For } \angle A = HG, \text{ Art. 10. Cor., } = EG - EH$$

$$= EG - (ED - DH) = EG + DH - ED;$$

but EG and DH are quadrants, or each subtends an angle $\frac{1}{2}\pi$;

$$\therefore \angle A = \pi - ED.$$

$$\text{Similarly,} \quad \angle B = \pi - EF, \quad \text{and} \quad \angle C = \pi - FD.$$

Again, $AB = AG - BG = AG - (IG - BI)$

$$= AG + BI - IG$$

$$= \tfrac{1}{2}\pi + \tfrac{1}{2}\pi - E, \text{ since } E = IG. \quad (10.\ \text{Cor.})$$

$$= \pi - E.$$

Similarly, $BC = \pi - F$, and $AC = \pi - D$.

19. *If a general equation be established between the sides and the angles of a spherical triangle, a true result is obtained if in the equation the supplements of the sides and of the angles respectively be written for the angles and sides which enter into the equation.*

For if the equation be proved for any triangle whatever, the sides a', b', c', and the angles A', B', C', of the *polar* triangle may be substituted in it, in the place of the sides a, b, c, and the angles A, B, C, of the *primitive* triangle. And in the equation as it then stands, putting for the sides and angles of the polar triangle their equivalents drawn from the primitive triangle, viz.

$$\pi - A = a', \quad \pi - B = b', \quad \pi - C = c',$$

$$\pi = a = A', \quad \pi - b = B', \quad \pi - c = C',$$

a true result is obtained; which differs from the original equation in having the supplements of the sides and of the angles respectively written for the angles and sides of the triangle.

Ex. If A, B, C be the angles, and a, b, c be the sides, of any triangle, it will hereafter be proved (29) that,

$$\operatorname{Cos} A = \frac{\cos a - \cos b \, . \cos c}{\sin b \, . \sin c}.$$

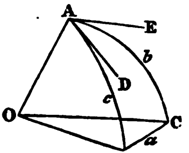
Therefore $$\operatorname{Cos} A' = \frac{\cos a' - \cos b' . \cos c'}{\sin b' . \sin c'}.$$

Now $A' = \pi - a, \quad a' = \pi - A, \quad b' = \pi - B, \quad c' = \pi - C;$

$$\therefore \operatorname{Cos} (\pi - a) = \frac{\cos(\pi - A) - \cos(\pi - B) . \cos(\pi - C)}{\sin(\pi - B) . \sin(\pi - C)}.$$

$$\therefore \operatorname{Cos} a = \frac{\cos A + \cos B . \cos C}{\sin B . \sin C}.$$

20. *Any two sides of a spherical triangle are together greater than the third; and the sum of the three sides is less than the circumference of a great circle.*



For, Eucl. XI. 20, any two of the angles AOB, BOC, COA,

$$\left(\text{or } \frac{\text{arc } AB}{AO},\ \frac{\text{arc } BC}{AO},\ \frac{\text{arc } CA}{AO}\right),$$

which form the solid angle at O, are together greater than the third. Therefore any two of the arcs AB, BC, CA are together greater than the third.

Also, Eucl. XI. 21, the three angles forming the solid angle at O are less than four right angles;

$$\therefore \frac{\text{arc } AB}{AO} + \frac{\text{arc } BC}{AO} + \frac{\text{arc } CA}{AO} < 2\pi;$$

$\therefore$ Arc AB + arc BC + arc $CA < 2\pi \,.\, AO$, which is the circumference of a circle whose radius is AO.

Cor. 1. Since the sum of the plane angles which contain *any* solid angle is less than four right angles, Eucl. XI. 21, it follows from the same mode of proof, that the sum of the sides of *any* polygonal figure which is described on a sphere, and whose sides are arcs of great circles, is less than the circumference of a great circle.

[The polygonal figure, however, must be such that the angle between any two adjacent sides is less than two right angles, and also that its area is contained on the surface of a hemisphere: for in the proof of the proposition of Euclid referred to, it is supposed that *all* the plane faces which contain the solid angle may be cut by *one* plane; which cannot be the case unless all the edges of the solid angle lie within the same hemisphere.]

Cor. 2. Also, let $ABCDE$ be a five-sided figure described on a sphere, and let it be divided into triangles by the arcs AC, AD. (A figure is easily drawn.)

$$\text{Then } AB + BC > AC;$$

$$\therefore AB + BC + CD > AC + CD;$$

$$\therefore AB + BC + CD > AD, \text{ for } AC + CD > AD;$$

$$\therefore AB + BC + CD + DE > AD + DE > AE.$$

And the same method of proof being applicable to a polygon of any number of sides, it follows that the sum of all the sides but one of a spherical polygon is greater than the remaining side.

Cor. 3. If a and b be two sides of a spherical triangle, since each is less than π,

$$\therefore\ a+b<2\pi, \qquad \therefore\ \tfrac{1}{2}(a+b)<\pi.$$

$$\text{And}\quad a-b<\pi, \qquad \therefore\ \tfrac{1}{2}(a-b)<\tfrac{1}{2}\pi.$$

Also, since any angle of a triangle is less than π;

$$\therefore\ A+B<2\pi, \qquad \therefore\ \tfrac{1}{2}(A+B)<\pi.$$

$$\text{And}\quad A-B<\pi, \qquad \therefore\ \tfrac{1}{2}(A-B)<\tfrac{1}{2}\pi.$$

Cor. 4. a', b', c', being the sides of the polar triangle,

$$a'+b'>c'; \qquad \therefore\ (\pi-A)+(\pi-B)>(\pi-C);$$

$$\therefore\ \pi>A+B-C; \qquad \text{or,}\quad A+B-C<\pi.$$

21. *The sum of the angles of a spherical triangle is greater than two right angles, and less than six.*

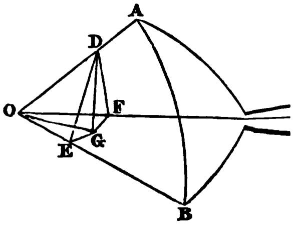
Let A, B, C be the angles, and a, b, c the sides, of a triangle; A', B', C', and a', b', c', the angles and sides of its polar triangle.

$$\begin{aligned} \text{Now}\quad 2\pi &> a'+b'+c'.\quad (20.) \\ &> (\pi-A)+(\pi-B)+(\pi-C) \\ &> 3\pi-(A+B+C); \\ &\therefore\ A+B+C>\pi. \end{aligned}$$

Again, since each of the angles A, B, C is less than π,

$$\therefore\ A+B+C<3\pi.$$

22. *The angles at the base of an isosceles triangle are equa[illegible] to each other.*



Suppose ABC to be an isosceles triangle, having $AB=AC$, and therefore $\angle\, AOB=\angle\, AOC$.

From D, any point in OA, draw DG perpendicular to the plane BOC; it is therefore at right angles to every line it meets in that plane; and from G draw GE and GF perpendicular to OB and OC; join ED, FD, OG.

Then $OE^2 = OG^2 - GE^2 = (OD^2 - DG^2) - (ED^2 - DG^2) = OD^2 - ED^2$;

$\therefore DE$ is perpendicular to OE, and $\therefore \angle DEG =$ inclination of the planes BOC and BOA, $= \angle B$.

Similarly, DF is perpendicular to OC, and $\angle DFG = \angle C$.

$$\text{Now } DE = OD \,.\, \sin AOB = OD \,.\, \sin AOC = DF.$$

$$\text{And } EG^2 = DE^2 - DG^2 = DF^2 - DG^2 = FG^2.$$

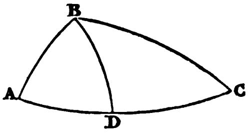
Hence, since GE, $ED = GF$, FD, each to each, and GD is common,

$$\therefore \angle DEG = \angle DFG, \quad \text{or } \angle B = \angle C.$$

23. Conversely, if $\angle B = \angle C$, or $\angle DEG = \angle DFG$, it may be shewn from the same figure that $AB = AC$; that is, the angles at the base being equal, the sides opposite to them are equal.

Cor. Hence also it follows that every equilateral triangle is also equiangular; and conversely, that every equiangular triangle is also equilateral.

24. *Of the two sides which are opposite to two unequal angles in a triangle, that is the greater which is opposite to the angle which is the greater.*



Let $\angle ABC$ be greater than $\angle BAC$. Then $AC > BC$.

Make $\angle ABD = \angle BAD$; $\therefore DA = DB$.

And $AC = CD + DA = CD + DB$;

But $CD + DB$ is $> BC$ (20); $\therefore AC > BC$.

25. Conversely, it may easily be shewn that of two angles in a triangle which are opposite to unequal sides, that is the greater which is opposite to the side which is the greater.

Cor. Hence $A - B$ and $a - b$ have the same sign.

26. Article 24 has been proved after the manner of Euclid I. 19. The following propositions may be enunciated for spherical triangles in the terms used for plane triangles, and may be proved in nearly the same words. Euclid I. Props. 4. 8. 24. 25. 26. &c.

27. To recapitulate the more important of the properties of Spherical Triangles which have been proved in this Chapter.

Art. 14.	1. A side must be less than a semicircle.
14.	2. An angle must be less than two right angles.
20.	3. Any two sides are together greater than the third.
24.	4. The greater side is opposite to the greater angle; and conversely.
25. Cor.	5. $A-B$ and $a-b$ are of the same sign.
20.	6. $a+b+c<2\pi$.
21.	7. $A+B+C>\pi$, and $<3\pi$.
20. Cor. 3.	8. $a+b<2\pi$, $a-b<\pi$.
20. Cor. 3.	9. $A+B<2\pi$, $A-B<\pi$.
20. Cor. 4.	10. $A+B-C<\pi$.

CHAPTER II.

FORMULÆ CONNECTING THE SIDES AND ANGLES OF A SPHERICAL TRIANGLE.

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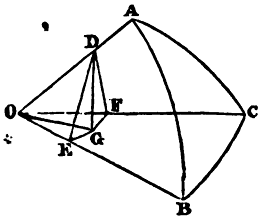
28. *The Sines of the Sides of a Spherical Triangle are as the Sines of the Angles to which they are respectively opposite.*



Let ABC be the triangle, O the centre of the sphere. From D, any point in OA, draw DG perpendicular to the plane BOC, and from G draw GE and GF perpendicular to OB and OC. Join DE, DF.

Then a line through G parallel to OB would be perpendicular to DG and GE, and ∴ to the plane DGE: wherefore, Euc. XI. 8, OE is perpendicular to the plane DEG, and ∴ to DE and EG; and ∴ $\angle DEG = \angle B$.

Similarly, $\angle DFG = \angle C$.

$$\text{Now } DE \,.\, \sin DEG = DG = DF \,.\, \sin DFG\,;$$

$$\therefore\ OD \,.\, \sin AOB \,.\, \sin DEG = OD \,.\, \sin AOC \,.\, \sin DFG\,;$$

$$\therefore\ \text{Sin}\, AB \,.\, \sin B = \sin AC \,.\, \sin C\,;$$

$$\therefore\ \frac{\text{Sin}\, B}{\text{Sin}\, AC} = \frac{\sin C}{\sin AB}.$$

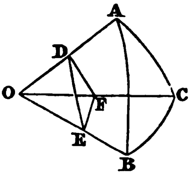
Similarly it may be proved that

$$\frac{\text{Sin}\, B}{\text{Sin}\, AC} = \frac{\text{Sin}\, A}{\text{Sin}\, BC}.$$

Wherefore

$$\frac{\text{Sin}\, A}{\text{Sin}\, BC} = \frac{\sin B}{\sin AC} = \frac{\sin C}{\sin AB} \quad \ldots\ldots\ldots\ldots\ldots (\text{i.})$$

29. *To express the Cosine of an Angle of a Triangle in terms of the Cosines and Sines of the Sides.*



From any point D in OA draw DE and DF, in the planes AOB and AOC, at right angles to AO. Therefore $\angle EDF$ is the inclination of those planes to each other, that is, the angle A. Join EF. Then, from the triangles FOE and FDE,

$$OF^2+OE^2-2\,OF.OE.\cos FOE=FE^2=FD^2+DE^2-2\,FD.DE.\cos FDE;$$

$$\therefore\ 2\,OF.OE.\cos FOE$$

$$=(OF^2-FD^2)+(OE^2-DE^2)+2\,FD.DE.\cos FDE$$

$$=2\,OD^2+2.FD.DE.\cos FDE;$$

$$\therefore\ \operatorname{Cos} FOE=\frac{OD}{OF}\cdot\frac{OD}{OE}+\frac{FD}{OF}\cdot\frac{DE}{OE}.\cos FDE;$$

$$\text{or,}\quad \operatorname{Cos} a=\cos b.\cos c+\sin b.\sin c.\cos A\ \ldots\ldots\ldots\ \text{(ii.)}$$

$$\text{Whence,}\quad \operatorname{Cos} A=\frac{\cos a-\cos b.\cos c}{\sin b.\sin c}.$$

Cor. By writing $\pi-a$ for A, &c. (19),

$$\operatorname{Cos} A+\cos B.\cos C=\sin B.\sin C.\cos a\ \ldots\ldots\ldots\ldots\ \text{(iii.)}$$

$$\text{Whence,}\quad \operatorname{Cos} a=\frac{\cos A+\cos B.\cos C}{\sin B.\sin C}.$$

30. *To shew that*

$$\operatorname{Cos} a.\sin b=\sin a.\cos b.\cos C+\sin c.\cos A.$$

Since by formula (ii),

$$\operatorname{Cos} c=\cos a.\cos b+\sin a.\sin b.\cos C,$$

$$\therefore\ \operatorname{Cos} a=\cos b.\cos c+\sin b.\sin c.\cos A,$$

$$=\cos b.(\cos a.\cos b+\sin a.\sin b.\cos C)+\sin b.\sin c.\cos A,$$

$$\therefore\ \cos a.(1-\cos^2 b)=\sin b.\{\sin a.\cos b.\cos C+\sin c.\cos A\};$$

$$\therefore\ \operatorname{Cos} a.\sin b=\sin a.\cos b.\cos C+\sin c.\cos A\ \ldots\ldots\ \text{(iv.)}$$

Cor. 1. By (iv), $\mathrm{Cos}\,a \,.\, \sin b = \sin a \,.\, \cos b \,.\, \cos C + \sin c \,.\, \cos A$,

$$\therefore \frac{\cos a}{\sin a} \,.\, \sin b = \cos b \,.\, \cos C + \frac{\sin c}{\sin a} \,.\, \cos A$$

$$= \cos b \,.\, \cos C + \frac{\sin C}{\sin A} \,.\, \cos A\,;$$

$$\therefore \mathrm{Cot}\,a \,.\, \sin b = \cos b \,.\, \cos C + \sin C \,.\, \cot A \,\ldots\ldots\ldots\, (\text{v.})$$

Cor. 2. By writing $\pi - a$ for A, &c., (19), the formula (iv.) becomes,

$$\mathrm{Sin}\,C \,.\, \cos a = \cos A \,.\, \sin B + \sin A \,.\, \cos B \,.\, \cos c \,\ldots\ldots\ldots\ldots\, (\text{vi.})$$

31. **Gauss' Theorem.**

If,

$$\begin{array}{ll} p = \cos\tfrac{1}{2}c \,.\, \sin\tfrac{1}{2}(A+B), & P = \cos\tfrac{1}{2}C \,.\, \cos\tfrac{1}{2}(a-b), \\ q = \cos\tfrac{1}{2}c \,.\, \cos\tfrac{1}{2}(A+B), & Q = \sin\tfrac{1}{2}C \,.\, \cos\tfrac{1}{2}(a+b), \\ r = \sin\tfrac{1}{2}c \,.\, \sin\tfrac{1}{2}(A-B), & R = \cos\tfrac{1}{2}C \,.\, \sin\tfrac{1}{2}(a-b), \\ s = \sin\tfrac{1}{2}c \,.\, \cos\tfrac{1}{2}(A-B), & S = \sin\tfrac{1}{2}C \,.\, \sin\tfrac{1}{2}(a+b), \end{array}$$

then the products p × q, p × r, p × s, q × r, q × s, r × s *are respectively equal to the products* P × Q, P × R, P × S, Q × R, Q × S, R × S.

I.
$$1 + \frac{\sin B}{\sin A} = 1 + \frac{\sin b}{\sin a}\,;$$

$$\therefore \sin a \,.\, (\sin A + \sin B) = \sin A \,.\, (\sin a + \sin b)\,;$$

$$\therefore \mathrm{Sin}\,c \,.\, (\sin A + \sin B) = \sin C \,.\, (\sin a + \sin b) \,\ldots\ldots\ldots\ldots\, (1).$$

Similarly, from $1 - \dfrac{\sin B}{\sin A} = 1 - \dfrac{\sin b}{\sin a}$, there would be got

$$\mathrm{Sin}\,c \,.\, (\sin A - \sin B) = \sin C \,.\, (\sin a - \sin b) \,\ldots\ldots\ldots\, (2).$$

II. By (iv.), $\mathrm{Sin}\,c \,.\, \cos A = \cos a \,.\, \sin b - \sin a \,.\, \cos b \,.\, \cos C$.

So also $\sin c \,.\, \cos B = \cos b \,.\, \sin a - \sin b \,.\, \cos a \,.\, \cos C$;

$$\therefore \mathrm{Sin}\,c \,.\, (\cos B + \cos A) = (\sin a \,.\, \cos b + \cos a \,.\, \sin b) \,.\, (1 - \cos C)$$

$$= \sin(a+b) \,.\, (1 - \cos C) \,\ldots\ldots\ldots\ldots\, (3).$$

Similarly, $\mathrm{Sin}\,c \,.\, (\cos B - \cos A) = \sin(a-b) \,.\, (1 + \cos C) \,\ldots\ldots\, (4).$

III. By writing $\pi - C$, &c. for c, &c. (19), and transposin terms of (3), and (4),

$$(1 + \cos c) . \sin (A + B) = \sin C . (\cos b + \cos a) \ldots\ldots\ldots\ldots$$

$$(1 - \cos c) . \sin (A - B) = \sin C . (\cos b - \cos a) \ldots\ldots\ldots\ldots$$

And expressing these six forms in terms of the sines and co of $\frac{1}{2} C$, $\frac{1}{2} c$, $\frac{1}{2} (A \pm B)$, $\frac{1}{2} (a \pm b)$, they respectively become,

$$\text{Sin} \tfrac{1}{2} c . \cos \tfrac{1}{2} c . \sin \tfrac{1}{2} (A+B) . \cos \tfrac{1}{2} (A-B) = \sin \tfrac{1}{2} C . \cos \tfrac{1}{2} C . \sin \tfrac{1}{2} (a+b) . \cos \tfrac{1}{2} (a$$

$$\text{Sin} \tfrac{1}{2} c . \cos \tfrac{1}{2} c . \sin \tfrac{1}{2} (A-B) . \cos \tfrac{1}{2} (A+B) = \sin \tfrac{1}{2} C . \cos \tfrac{1}{2} C . \sin \tfrac{1}{2} (a-b) . \cos \tfrac{1}{2} (a$$

$$\text{Sin} \tfrac{1}{2} c . \cos \tfrac{1}{2} c . \cos \tfrac{1}{2} (A+B) . \cos \tfrac{1}{2} (A-B) = \sin \tfrac{1}{2} C . \sin \tfrac{1}{2} C . \sin \tfrac{1}{2} (a+b) . \cos \tfrac{1}{2} (a$$

$$\text{Sin} \tfrac{1}{2} c . \cos \tfrac{1}{2} c . \sin \tfrac{1}{2} (A+B) . \sin \tfrac{1}{2} (A-B) = \cos \tfrac{1}{2} C . \cos \tfrac{1}{2} C . \sin \tfrac{1}{2} (a-b) . \cos \tfrac{1}{2} (a$$

$$\text{Cos} \tfrac{1}{2} c . \cos \tfrac{1}{2} c . \sin \tfrac{1}{2} (A+B) . \cos \tfrac{1}{2} (A+B) = \sin \tfrac{1}{2} C . \cos \tfrac{1}{2} C . \cos \tfrac{1}{2} (a+b) . \cos \tfrac{1}{2} (a$$

$$\text{Sin} \tfrac{1}{2} c . \sin \tfrac{1}{2} c . \sin \tfrac{1}{2} (A-B) . \cos \tfrac{1}{2} (A-B) = \sin \tfrac{1}{2} C . \cos \tfrac{1}{2} C . \sin \tfrac{1}{2} (a+b) . \sin \tfrac{1}{2} (a$$

which six equations are,

$$\left\{\begin{matrix} p \times s = P \times S \\ q \times r = Q \times R \end{matrix}\right\} \ldots\ldots\ldots \left\{\begin{matrix} q \times s = Q \times S \\ p \times r = P \times R \end{matrix}\right\} \ldots\ldots\ldots \left\{\begin{matrix} p \times q = P \times \\ r \times s = R \times \end{matrix}\right.$$

32. *The quantities* p, q, r, s *are respectively equal tc* Q, R, S.

$$\text{Since } pq . pr = PQ . PR ;$$

$$\therefore p^2 . qr = P^2 . QR ;$$

$$\therefore p^2 = P^2 .$$

$$\text{Similarly, } \quad q^2 = Q^2, \quad r^2 = R^2, \quad s^2 = S^2 ;$$

$$\therefore p = \pm P, \quad q = \pm Q, \quad r = \pm R, \quad s = \pm S.$$

Now in a spherical triangle, (27),

$\frac{1}{2} (A + B)$ and $\frac{1}{2} (a + b)$, are each less than π;

and $\frac{1}{2} (A - B)$ and $\frac{1}{2} (a - b)$, are each less than $\frac{1}{2} \pi$, and of the same sign;

and $\frac{1}{2} C$ and $\frac{1}{2} c$ are each less than $\frac{1}{2} \pi$;

$\therefore p, r, s, P, R, S$ are all positive quantities.

Also since $pq = PQ$, and $p = +P$, $\therefore q = +Q$;

$\therefore p = +P$, or $\text{Cos}\,\tfrac{1}{2}c \,.\, \sin\tfrac{1}{2}(A+B) = \cos\tfrac{1}{2}C \,.\, \cos\tfrac{1}{2}(a-b)$....(vii.)

$q = +Q$, $\text{Cos}\,\tfrac{1}{2}c \,.\, \cos\tfrac{1}{2}(A+B) = \sin\tfrac{1}{2}C \,.\, \cos\tfrac{1}{2}(a+b)$... (viii.)

$r = +R$, $\text{Sin}\,\tfrac{1}{2}c \,.\, \sin\tfrac{1}{2}(A-B) = \cos\tfrac{1}{2}C \,.\, \sin\tfrac{1}{2}(a-b)$ .... (ix.)

$s = +S$, $\text{Sin}\,\tfrac{1}{2}c \,.\, \cos\tfrac{1}{2}(A-B) = \sin\tfrac{1}{2}C \,.\, \sin\tfrac{1}{2}(a+b)$..... (x.)

These formulæ are practically of great use in the investigations of Plane Astronomy.

Cor. From (viii.) it appears that $\text{Cos}\,\tfrac{1}{2}(A+B)$ and $\text{Cos}\,\tfrac{1}{2}(a+b)$ are of the same sign, and therefore $\tfrac{1}{2}(A+B)$ and $\tfrac{1}{2}(a+b)$ are both greater or both less than $\tfrac{1}{2}\pi$.

Def. When two angles are both greater or both less than a right angle, they are said to be *of the same*, or *of like affection.*

Thus $\tfrac{1}{2}(A+B)$ and $\tfrac{1}{2}(a+b)$ are of like affection; a property of spherical triangles which may be added to those enumerated in (27).

33. Since

$$\frac{p}{q} = \frac{P}{Q}; \quad \therefore \text{Tan}\,\tfrac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)} \,.\, \cot\tfrac{1}{2}C\ldots\ldots\text{(xi.)}$$

$$\frac{r}{s} = \frac{R}{S}; \quad \therefore \text{Tan}\,\tfrac{1}{2}(A-B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)} \,.\, \cot\tfrac{1}{2}C\ldots\ldots\text{(xii.)}$$

$$\frac{S}{Q} = \frac{s}{q}; \quad \therefore \text{Tan}\,\tfrac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)} \,.\, \tan\tfrac{1}{2}c\ldots\ldots\text{(xiii.)}$$

$$\frac{R}{P} = \frac{r}{p}; \quad \therefore \text{Tan}\,\tfrac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)} \,.\, \tan\tfrac{1}{2}c\ldots\ldots\text{(xiv.)}$$

These four equations are called Napier's Analogies, and are of great use in the solution of triangles*.

---

* The quantities p, q, r, s, P, Q, R, S, will be most easily remembered from Napier's first and second Analogies written thus,

$$\frac{p}{q} = \frac{\sin\frac{1}{2}(A+B) \,.\, \cos\frac{1}{2}c}{\cos\frac{1}{2}(A+B) \,.\, \cos\frac{1}{2}c} = \frac{\cos\frac{1}{2}(a-b) \,.\, \cos\frac{1}{2}C}{\cos\frac{1}{2}(a+b) \,.\, \sin\frac{1}{2}C} = \frac{P}{Q}.$$

$$\frac{r}{s} = \frac{\sin\frac{1}{2}(A-B) \,.\, \sin\frac{1}{2}c}{\cos\frac{1}{2}(A-B) \,.\, \sin\frac{1}{2}c} = \frac{\sin\frac{1}{2}(a-b) \,.\, \cos\frac{1}{2}C}{\sin\frac{1}{2}(a+b) \,.\, \sin\frac{1}{2}C} = \frac{R}{S}.$$

34. The following expressions can be proved, without much difficulty, from the formulæ of (32); they may, however, be proved independently of Gauss' Theorem from (i), (ii), (iii).

To find the values of Cos ½ A, Sin ½ A, Tan ½ A, *and* Sin A, *in terms of* a, b, c.

By (ii), $\sin b \,.\, \sin c \,.\, \cos A = \cos a - \cos b \,.\, \cos c$;

$$\therefore\ \sin b \,.\, \sin c \,.\, (1 + \cos A) = \cos a - (\cos b \,.\, \cos c - \sin b \,.\, \sin c);$$

$$\therefore\ \sin b \,.\, \sin c \,.\, 2 \cos^2 \tfrac{1}{2} A = \cos a - \cos (b + c)$$

$$= 2 \sin \tfrac{1}{2} (a + b + c) \,.\, \sin \tfrac{1}{2} (b + c - a).$$

And if $S = \frac{1}{2}(a + b + c)$, $S - a = \frac{1}{2}(a + b + c) - a = \frac{1}{2}(b + c - a)$,

$$\therefore\ \cos^2 \tfrac{1}{2} A = \frac{\sin S \,.\, \sin (S - a)}{\sin b \,.\, \sin c};$$

$$\therefore\ \text{Cos}\, \tfrac{1}{2} A = \sqrt{\frac{\sin S \,.\, \sin (S - a)}{\sin b \,.\, \sin c}} \ldots\ldots\ldots \text{(xv.)}$$

$$\text{Again,}\ 2 \sin^2 \tfrac{1}{2} A = 1 - \cos A = 1 - \frac{\cos a - \cos b \,.\, \cos c}{\sin b \,.\, \sin c};$$

$$\text{Whence,}\ \text{Sin}\, \tfrac{1}{2} A = \sqrt{\frac{\sin (S - b) \,.\, \sin (S - c)}{\sin b \,.\, \sin c}} \ldots\ldots \text{(xvi.)}$$

$$\text{Also,}\ \text{Tan}\, \tfrac{1}{2} A = \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} = \sqrt{\frac{\sin (S - b) \,.\, \sin (S - c)}{\sin S \,.\, \sin (S - a)}} \ldots\ldots \text{(xvii.)}$$

And $\text{Sin}\, A = 2 \sin \frac{1}{2} A \,.\, \cos \frac{1}{2} A$

$$= \frac{2}{\sin b \,.\, \sin c} \sqrt{\{\sin S \,.\, \sin (S - a) \,.\, \sin (S - b) \,.\, \sin (S - c)\}} \,.. \text{(xviii.)}$$

The positive signs of the square roots are taken in these cases, because $\frac{1}{2} A$ is necessarily less than $\frac{1}{2} \pi$.

35. *To find the values of* Cos ½ a, Sin ½ a, Tan ½ a, *and* Sin a, *in terms of* A, B, C.

By (iii), $$\operatorname{Cos} a = \frac{\cos A + \cos B \cdot \cos C}{\sin B \cdot \sin C};$$

$$\therefore\ 2\cos^2 \tfrac{1}{2}a = 1 + \cos a = \frac{\cos A + \cos B \cdot \cos C + \sin B \cdot \sin C}{\sin B \cdot \sin C}$$

$$= \frac{\cos A + \cos (B - C)}{\sin B \cdot \sin C} = \frac{2\cos \tfrac{1}{2}(A + C - B) \cdot \cos \tfrac{1}{2}(A + B - C)}{\sin B \cdot \sin C}.$$

And if $S' = \frac{1}{2}(A + B + C)$, then $S' - B = \frac{1}{2}(A + C - B)$, &c.;

$$\therefore\ \operatorname{Cos} \tfrac{1}{2}a = \sqrt{\frac{\cos (S' - B) \cdot \cos (S' - C)}{\sin B \cdot \sin C}} \ldots\ldots \text{(xix.)}$$

Again, $$2\sin^2 \tfrac{1}{2}a = 1 - \cos a = 1 - \frac{\cos A + \cos B \cdot \cos C}{\sin B \cdot \sin C}$$

$$= -\frac{\cos A + \cos (B + C)}{\sin B \cdot \sin C};$$

Whence, $$\operatorname{Sin} \tfrac{1}{2}a = \sqrt{\frac{-\cos S' \cdot \cos (S' - A)}{\sin B \cdot \sin C}} \ldots\ldots\ldots\ldots \text{(xx.)}$$

Also, $$\operatorname{Tan} \tfrac{1}{2}a = \frac{\sin \tfrac{1}{2}a}{\cos \tfrac{1}{2}a} = \sqrt{\frac{-\cos S' \cdot \cos (S' - A)}{\cos (S' - B) \cdot \cos (S' - C)}} \ldots\ldots \text{(xxi.)}$$

And $$\operatorname{Sin} a = 2\sin \tfrac{1}{2}a \cdot \cos \tfrac{1}{2}a$$

$$= \frac{2}{\sin B \cdot \sin C}\sqrt{\{-\cos S' . \cos (S' - A) \cdot \cos (S' - B) \cdot \cos (S' - C)\}} \text{ (xxii.)}$$

NOTE. Since the angles of a spherical triangle are together greater than two right angles and less than six, S' is greater than one right angle and less than three, and its cosine is therefore a negative quantity. Also since (27) $A + B - C < \pi$, therefore $\cos \frac{1}{2}(A + B - C)$, or $\cos (S' - C)$, is a positive quantity; and in like manner $\cos (S' - B)$ and $\cos (S' - A)$ are positive. Wherefore the last three formulæ are not *imaginary* quantities, as at first sight they appear to be, but *real* quantities, as they ought to be.

36. *To prove Napier's Analogies from the formulæ* (i), (ii), (iii).

By (iii), $\operatorname{Cos} A + \cos B \,.\, \cos C = \cos a \,.\, \sin B \,.\, \sin C.$

So, $\operatorname{Cos} B + \cos A \,.\, \cos C = \cos b \,.\, \sin A \,.\, \sin C.$

By addition,

$$(\cos A + \cos B)\,.\,(1 + \cos C) = (\cos a \,.\, \sin B + \cos b \,.\, \sin A)\,.\, \sin C;$$

$$\therefore\ (\cos A + \cos B)\,.\, 2\cos^2 \tfrac{1}{2} C$$

$$= \left(\cos a \,.\, \sin A \,.\, \frac{\sin b}{\sin a} + \cos b \,.\, \sin A\right).\, 2 \sin \tfrac{1}{2} C \,.\, \cos \tfrac{1}{2} C;$$

$$\therefore\ \cos A + \cos B = \frac{\sin A}{\sin a}\,.\,(\cos a \,.\, \sin b + \cos b \,.\, \sin a)\,.\, \tan \tfrac{1}{2} C$$

$$= \frac{\sin A}{\sin a}\,.\, \sin (a + b)\,.\, \tan \tfrac{1}{2} C.$$

Again, $\operatorname{Sin} A + \sin B = \sin A \,.\, \left(1 + \dfrac{\sin B}{\sin A}\right) = \sin A \,.\, \left(1 + \dfrac{\sin b}{\sin a}\right)$

$$= \frac{\sin A}{\sin a}\,.\,(\sin a + \sin b);$$

$$\therefore\ \frac{\sin A + \sin B}{\cos A + \cos B} = \frac{\sin a + \sin b}{\sin (a + b)}\,.\, \cot \tfrac{1}{2} C,$$

$$\frac{2 \,.\, \sin \frac{1}{2}(A + B)\,.\, \cos \frac{1}{2}(A - B)}{2 \,.\, \cos \frac{1}{2}(A + B)\,.\, \cos \frac{1}{2}(A - B)} = \frac{2 \,.\, \sin \frac{1}{2}(a + b)\,.\, \cos \frac{1}{2}(a - b)}{2 \,.\, \sin \frac{1}{2}(a + b)\,.\, \cos \frac{1}{2}(a + b)}\,.\, \cot \tfrac{1}{2} C;$$

$$\therefore\ \operatorname{Tan} \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)}\,.\, \cot \tfrac{1}{2} C \ldots\ldots\ldots\ldots(\alpha).$$

In like manner, $\dfrac{\operatorname{Sin} A - \sin B}{\operatorname{Cos} A + \cos B} = \dfrac{\sin a - \sin b}{\sin (a + b)}\,.\, \cot \tfrac{1}{2} C;$

Whence, $\operatorname{Tan} \tfrac{1}{2}(A - B) = \dfrac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)}\,.\, \cot \tfrac{1}{2} C \ldots\ldots\ldots\ldots(\beta).$

By writing $\pi - A$ for a, &c., (19), the two formulæ (α) and (β) become

$$\operatorname{Tan} \tfrac{1}{2} (a+b) = \frac{\cos \frac{1}{2} (A-B)}{\cos \frac{1}{2} (A+B)} \cdot \tan \tfrac{1}{2} c \ldots\ldots\ldots\ldots (\gamma).$$

$$\operatorname{Tan} \tfrac{1}{2} (a-b) = \frac{\sin \frac{1}{2} (A-B)}{\sin \frac{1}{2} (A+B)} \cdot \tan \tfrac{1}{2} c \ldots\ldots\ldots\ldots (\delta).$$

These last two analogies may be easily proved independently by beginning with the formula

$$\operatorname{Cos} a = \cos b \, . \cos c + \sin b \, . \sin c \, . \cos A.$$

CHAPTER III.

ON THE SOLUTION OF RIGHT-ANGLED TRIANGLES.

37. RIGHT-ANGLED triangles may in all cases, (with an exception which will be pointed out, Art. 40.), be solved by means of the following formulæ, when, besides the right angle, two other quantities are given out of the three sides and the two remaining angles.

If A, B, C be the angles of a spherical triangle, and a, b, c the sides respectively opposite to them, it has been proved that

(ii.) $\text{Cos}\, c = \cos a \,.\, \cos b + \sin a \,.\, \sin b \,.\, \cos C$ ....(1)

(iii.) $\text{Cos}\, c \,.\, \sin A \,.\, \sin B = \cos C + \cos A \,.\, \cos B$ ....(2)

$\text{Cos}\, a \,.\, \sin B \,.\, \sin C = \cos A + \cos B \,.\, \cos C$ ....(3)

(i.) $\text{Sin}\, a \,.\, \sin C = \sin c \,.\, \sin A$ ....(4)

(v.) $\text{Cot}\, a \,.\, \sin b = \cos b \,.\, \cos C + \sin C \,.\, \cot A$ ....(5)

$\text{Cot}\, c \,.\, \sin a = \cos a \,.\, \cos B + \sin B \,.\, \cot C$ .... (6)

By making $C = 90^0$, there will be obtained,

From (1), $\text{Cos}\, c = \cos a \,.\, \cos b$.

(2), $\text{Cos}\, c = \cot A \,.\, \cot B$.

(3), $\text{Cos} A = \cos a \,.\, \sin B$.

$\text{Cos} B = \cos b \,.\, \sin A$.

(4), $\text{Sin}\, a = \sin c \,.\, \sin A$.

$\text{Sin}\, b = \sin c \,.\, \sin B$.

(5), $\text{Sin}\, b = \tan a \,.\, \cot A$.

$\text{Sin}\, a = \tan b \,.\, \cot B$.

(6), $\text{Cos} B = \cot c \,.\, \tan a$.

$\text{Cos} A = \cot c \,.\, \tan b$.

38. **Napier's Rules.** The formulæ of the last Article are comprised under two Rules, which take their name from Napier, who first gave them.

The right angle being left out of consideration, the *two sides which include the right angle*, and the *complements of the hypothenuse* and *of the other angles*, are called the *Circular Parts* of the triangle. Any one of these being fixed upon as *the middle part* (M), the two circular parts next to it and *immediately* joining it are called *the adjacent parts* (A_1, A_2), and the other two parts are called *the opposite parts*, (O_1, O_2).

Thus in the triangle ABC whose right angle is C;

If M be	The adjacent Parts are	The opposite Parts are
a, (one of the sides including the right angle).	$\frac{1}{2}\pi - B$, b.	$\frac{1}{2}\pi - A$, $\frac{1}{2}\pi - c$.
$\frac{1}{2}\pi - A$, (the complement of an angle).	b, $\frac{1}{2}\pi - c$.	$\frac{1}{2}\pi - B$, a.
$\frac{1}{2}\pi - c$, (the complement of the hypothenuse).	$\frac{1}{2}\pi - A$, $\frac{1}{2}\pi - B$.	a, b.

And **Napier's Rules** are,

Sin M = *product of the* Tangents *of the* Adjacent *parts* = $\tan A_1 . \tan A_2$,

Sin M = *product of the* Cosines *of the* Opposite *parts* = $\cos O_1 . \cos O_2$,

with which the formulæ of the last Article will, on trial, be found to agree.

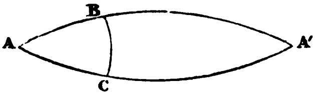
39. *To shew that in a triangle* ABC *in which* C *is a right angle*, A *and* a *are of the same affection*, *as are also* B *and* b.

$$\sin b = \tan\left(\tfrac{1}{2}\pi - A\right) . \tan a = \cot A . \tan a = \frac{\tan a}{\tan A}.$$

Now since b is less than π, $\sin b$ is positive; therefore $\tan a$ and $\tan A$ must be of the same sign. And because π is the limit both of a and of A, these angles must be both greater or both less than a right angle; that is, A and a must be of the same affection.

Similarly, from $\sin a = \dfrac{\tan b}{\tan B}$ it appears that B and b are of the same affection.

40. *If, in a right-angled triangle, an Angle and the Side opposite to it be the only quantities given, the triangle cannot be determined.*



For if the circles AB and AC intersect again in A', and C be a right angle, it is evident that ACB and $A'CB$ have the angles A, A' equal, and CB, the side opposite to these angles, is the same in both triangles. It is therefore ambiguous whether ABC or $A'BC$ be the triangle sought.

This ambiguity will also be found to exist, if it be attempted to determine the triangle by Napier's rules. For it cannot be determined from the equation

$$\operatorname{Sin} AC = \tan CB \,.\, \tan\left(\tfrac{1}{2}\pi - A\right)$$

whether the angle AC is to be taken, or its supplement $A'C$.

41. The solutions of the other cases of a right-angled triangle from two given parts are not ambiguous, if attention be paid to these two principles,

(1) The greater side is opposite to the greater angle.

(2) An angle and the side opposite to it are of the same affection.

[For example: Let c and A be given, to find a, B, b.

$$\text{Now}\quad \operatorname{Sin} a = \cos\left(\tfrac{1}{2}\pi - A\right) . \cos\left(\tfrac{1}{2}\pi - c\right) = \sin A \,.\, \sin c\,;$$

and since a and A are of the same affection (39), the greater or lesser angle which satisfies this equation is to be taken for a, according as A is greater or less than $\frac{1}{2}\pi$.

$$\text{Again,}\quad \operatorname{Sin}\left(\tfrac{1}{2}\pi - c\right),\ \text{ or } \cos c, = \cot A \,.\, \cot B\,;$$

$$\therefore\ \operatorname{Tan} B = \cot A \,.\, \sec c.$$

And B is $< 90^\circ$, or $> 90^\circ$, according as the second member of the equation is positive or negative; that is, as A and c are of like or unlike affection.

$$\text{Again,}\quad \operatorname{Sin}\left(\tfrac{1}{2}\pi - c\right),\ \text{ or } \cos c, = \cos a \,.\, \cos b\,;$$

$$\therefore\ \operatorname{Cos} b = \cos c \,.\, \sec a.$$

And b is $< 90^\circ$, or $> 90^\circ$, according as the second member of the equation is positive or negative; that is, as a and c are of like or unlike affection.]

42. In selecting a formula, attention must be paid to the principles laid down in Appendix II. to Pl. Trig. The following formulæ may be used with advantage, when the side or angle required is small, or nearly equal to one right angle, or to two.

By Napier's rules, $\operatorname{Cos} c = \cot A \,.\, \cot B$, whence c cannot be accurately determined, if it be either a very small angle or nearly equal to two right angles.

$$\text{Now}\quad 2\sin^2 \tfrac{1}{2}c = 1 - \cos c = 1 - \cot A \,.\, \cot B = 1 - \frac{\cos A \,.\, \cos B}{\sin A \,.\, \sin B};$$

$$\therefore\ \operatorname{Sin} \tfrac{1}{2}c = \sqrt{\frac{-\cos(A+B)}{2\sin A \,.\, \sin B}}.^{*}$$

$$\text{So,}\quad \operatorname{Cos} \tfrac{1}{2}c = \sqrt{\tfrac{1}{2}(1+\cos c)} = \sqrt{\frac{\cos(A-B)}{2\sin A \,.\, \sin B}}.^{*}$$

In like manner there is obtained from $\operatorname{Cos} a = \dfrac{\sin(\frac{1}{2}\pi - A)}{\sin B}$,

$$\left\{\begin{aligned} \operatorname{Cos} \tfrac{1}{2}a &= \sqrt{\frac{\sin\{\frac{1}{2}(B-A)+\frac{1}{4}\pi\} \,.\, \cos\{\frac{1}{2}(B+A)-\frac{1}{4}\pi\}}{\sin B}}; \\ \operatorname{Sin} \tfrac{1}{2}a &= \sqrt{\frac{\sin\{\frac{1}{2}(B+A)-\frac{1}{4}\pi\} \,.\, \cos\{\frac{1}{2}(B-A)+\frac{1}{4}\pi\}}{\sin B}}. \end{aligned}\right.$$

So from $\operatorname{Cos} A = \cot c \,.\, \tan b$,

$$\left\{\begin{aligned} \operatorname{Sin} \tfrac{1}{2}A &= \sqrt{\tfrac{1}{2}(1 - \cot c \,.\, \tan b)} = \sqrt{\frac{\sin(c-b)}{2\sin c \,.\, \cos b}}, \\ \operatorname{Cos} \tfrac{1}{2}A &= \sqrt{\tfrac{1}{2}(1 + \cot c \,.\, \tan b)} = \sqrt{\frac{\sin(c+b)}{2\sin c \,.\, \cos b}}. \end{aligned}\right.$$

When c and A are given, if a be nearly a right angle, it cannot be accurately determined from its sine. In this case $\operatorname{Cot} B = \cos c \,.\, \tan A$ determines B, and a may be found from the formula for $\operatorname{Sin} \frac{1}{2}a$, or from that for $\operatorname{Cos} \frac{1}{2}a$. (35).

---

[*] Since (21), $A + B + C > \pi$, $\therefore$ if $C = \frac{1}{2}\pi$, $A + B > \frac{1}{2}\pi$,

and since (20. Cor. 4), $A + B - C < \pi$, $\therefore$ .... $A + B < \frac{3}{2}\pi$;

Wherefore, $\operatorname{Cos}(A + B)$ is necessarily a negative quantity here.

Also since (20. Cor. 4), $A + C - B < \pi$, $\therefore$ $A - B < \frac{1}{2}\pi$, and $\cos(A - B)$ is positive.

43. Def. A triangle is called *Quadrantal*, if any one of it
sides be a quadrant.

A quadrantal triangle ABC, *of which the side* c *is*
quadrant, may be solved by Napier's Rules, if the quadranta
side be neglected, and $\frac{1}{2}\pi - a$, $\frac{1}{2}\pi - b$, A, B, $C - \frac{1}{2}\pi$, *be take*
for "the circular parts." (38).

Let $A'B'C'$ be the polar triangle. It may be solved by Napier
Rules, because $C'\,(=\pi - c = \frac{1}{2}\pi)$ is a right angle;

$$\therefore \left\{\begin{array}{lll} \cos c' = \cot A' . \cot B', & \text{or} & = \cos a' . \cos b', \\ \sin a' = \tan b' . \cot B', & \text{or} & = \sin A' . \sin c', \\ \cos A' = \tan b' . \cot c', & \text{or} & = \cos a' . \sin B'. \end{array}\right.$$

Substituting $\pi - A$, $\pi - a$, &c. for A', a', &c. these equations
become,

$$\left\{\begin{array}{lll} -\cos C = \cot a . \cot b, & \text{or} & = \cos A . \cos B, \\ \sin A = \tan B . \cot b, & \text{or} & = \sin a . \sin C, \\ -\cos a = \tan B . \cot C, & \text{or} & = -\cos A . \sin b; \end{array}\right.$$

which may be represented thus,

$$\left\{\begin{array}{ll} \text{Sin}\,(C - \frac{1}{2}\pi) = \tan(\frac{1}{2}\pi - a) . \tan(\frac{1}{2}\pi - b), & \text{or} = \cos A . \cos B, \\ \text{Sin}\,A = \tan B . \tan(\frac{1}{2}\pi - b), & \text{or} = \cos(\frac{1}{2}\pi - a) . \cos(C - \frac{1}{2}\pi), \\ \text{Sin}\,(\frac{1}{2}\pi - a) = \tan B . \tan(C - \frac{1}{2}\pi), & \text{or} = \cos A . \cos(\frac{1}{2}\pi - b); \end{array}\right.$$

which was to be proved.

A Collection of Examples for practice is added at the end of this Treatise.

CHAPTER IV.

ON THE SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

44. *Let the three Sides be given.* (a, b, c).

The *Angles* may be determined from one of the formulæ (xv.), (xvi.), (xvii.), (xviii.).

45. *Let the three Angles be given.* (A, B, C).

The *Sides* may be determined from one of the formulæ (xix.), (xx.), (xxi.), (xxii.).

46. *Let two Sides and the included Angle be given.* (a, C, b).

By Napier's first and second Analogies, (xi.) and (xii.),

$$\left\{\begin{aligned} \text{Tan}\,\tfrac{1}{2}(A+B) &= \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\,.\cot\tfrac{1}{2}C,\\ \text{Tan}\,\tfrac{1}{2}(A-B) &= \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\,.\cot\tfrac{1}{2}C,\end{aligned}\right.$$

$\frac{1}{2}(A+B)$, and $\frac{1}{2}(A-B)$, are determined;

$$\therefore \left\{\begin{aligned} A &= \tfrac{1}{2}(A+B)+\tfrac{1}{2}(A-B),\\ B &= \tfrac{1}{2}(A+B)-\tfrac{1}{2}(A-B),\end{aligned}\right\} \text{ are known.}$$

And A and B being known, c is found from

$$\text{Sin}\, c = \sin a\,.\,\frac{\sin C}{\sin A}\,.\text{*}$$

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* The easiest practical method of solving this case is by letting fall a perpendicular on AC from B, cutting it in D, and so dividing the triangle into two right-angled triangles.

$$\text{Then}\quad \text{Tan}\,CD = \cos C\,.\,\tan a, \text{ which gives } CD.$$

$$\text{And}\quad \therefore\ AD, = b - CD, \text{ is known.}$$

$$\text{Also}\quad \text{Cos}\, c = \cos AD\,.\cos DB = \cos AD\,.\,\frac{\cos a}{\cos CD}\,.$$

47. *To determine* c *independently of* A *and* B, *by forms adapted to logarithmic computation.*

$$\text{Cos}\, c = \cos a \,.\, \cos b + \sin a \,.\, \sin b \,.\, \cos C$$

$$= \cos b \,.\, (\cos a + \sin a \,.\, \tan b \,.\, \cos C).$$

Let θ be an angle such that $\quad \text{Tan}\, \theta = \tan b \,.\, \cos C \ldots\ldots\ldots\ldots(1).$

$$\text{Then} \quad \text{Cos}\, c = \cos b \,.\, \left(\cos a + \sin a \,.\, \frac{\sin \theta}{\cos \theta}\right)$$

$$= \frac{\cos b}{\cos \theta} \,.\, \cos (a - \theta) \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1), $L \tan \theta = L \tan b + L \cos C - 10$; which gives θ.

(2), $L \cos c = L \cos b + L \cos (a - \theta) - L \cos \theta$; which gives c.

48. *Let two Angles and the included Side be given.* (A, c, B)

From Napier's third and fourth Analogies, (xiii.) and (xiv.),

$$\begin{cases} \text{Tan}\, \tfrac{1}{2}(a + b) = \dfrac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \,.\, \tan \tfrac{1}{2} c, \\[2ex] \text{Tan}\, \tfrac{1}{2}(a - b) = \dfrac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \,.\, \tan \tfrac{1}{2} c, \end{cases}$$

$\frac{1}{2}(a + b)$ and $\frac{1}{2}(a - b)$ are determined;

$$\therefore \left.\begin{cases} a = \tfrac{1}{2}(a + b) + \tfrac{1}{2}(a - b), \\ b = \tfrac{1}{2}(a + b) - \tfrac{1}{2}(a - b), \end{cases}\right\} \text{are known.}$$

And a and b being known, C is found from

$$\text{Sin}\ C = \sin A \,.\, \frac{\sin c}{\sin a}\,.$$

49. *To find* C *independently of* a *and* b, *by forms adapted to logarithmic computation.*

$$\text{Cos}\ C = \cos c \,.\, \sin A \,.\, \sin B - \cos A \,.\, \cos B\,;$$

$$\therefore \quad 2 \cos^2 \tfrac{1}{2} C - 1 = (1 - 2 \sin^2 \tfrac{1}{2} c) \,.\, \sin A \,.\, \sin B - \cos A \,.\, \cos B$$

$$= - \cos (A + B) - 2 \sin^2 \tfrac{1}{2} c \,.\, \sin A \,.\, \sin B\,;$$

$$\therefore\ 2\cos^2\tfrac{1}{2}C = 1 - \cos(A+B) - 2\sin^2\tfrac{1}{2}c \,.\, \sin A \,.\, \sin B;$$

$$\therefore\ \cos^2\tfrac{1}{2}C = \sin^2\tfrac{1}{2}(A+B) - \sin^2\tfrac{1}{2}c \,.\, \sin A \,.\, \sin B.$$

Now $\text{Sin}^2\tfrac{1}{2}c \,.\, \sin A \,.\, \sin B$ is necessarily positive, and less than unity; wherefore there is an angle θ such that

$$\text{Sin}^2\theta = \sin^2\tfrac{1}{2}c \,.\, \sin A \,.\, \sin B \ldots\ldots\ldots\ldots\ldots\ldots(1);$$

$$\therefore\ \text{Cos}^2\tfrac{1}{2}C = \sin^2\tfrac{1}{2}(A+B) - \sin^2\theta$$

$$= \sin\{\tfrac{1}{2}(A+B)+\theta\} \,.\, \sin\{\tfrac{1}{2}(A+B)-\theta\}.\quad \text{Pl. Trig. Art. 54}\ldots.(2).$$

From (1), $L\sin\theta = \tfrac{1}{2}(L\sin A + L\sin B) + L\sin\tfrac{1}{2}c - 10$; which gives θ.

From (2), $L\cos\tfrac{1}{2}C = \tfrac{1}{2}[L\sin\{\tfrac{1}{2}(A+B)+\theta\} + L\sin\{\tfrac{1}{2}(A+B)-\theta\}]$; which gives C.

50. *Let two Angles and a Side opposite to one of them be given.* (A, B, a).

$$\text{Sin}\, b = \sin a \,.\, \frac{\sin B}{\sin A}.$$

From (xi.), $$\text{Tan}\tfrac{1}{2}C = \frac{\cos\tfrac{1}{2}(a-b)}{\cos\tfrac{1}{2}(a+b)} \,.\, \cot\tfrac{1}{2}(A+B).$$

From (xiii.), $$\text{Tan}\tfrac{1}{2}c = \frac{\cos\tfrac{1}{2}(A+B)}{\cos\tfrac{1}{2}(A-B)} \,.\, \tan\tfrac{1}{2}(a+b).$$

And b having been determined from the first of these equations, C and c may be found from the others.

51. *To determine* C *and* c *independently of* b, *by forms adapted to logarithmic computation.*

$$\text{Cos}\, A = \cos a \,.\, \sin B \,.\, \sin C - \cos B \,.\, \cos C$$

$$= \cos B \,.\, (\cos a \,.\, \tan B \,.\, \sin C - \cos C).$$

Let θ be an angle such that $\text{Cot}\,\theta = \cos a \,.\, \tan B \ldots\ldots\ldots\ldots(1)$;

$$\therefore\ \text{Cos}\, A = \cos B \,.\left(\frac{\cos\theta}{\sin\theta}\,.\sin C - \cos C\right)$$

$$= \frac{\cos B}{\sin\theta}\,.\sin(C-\theta)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2).$$

From (1), $L\cot\theta = L\cos a + L\tan B - 10$; which gives θ.

(2), $L\sin(C-\theta) = L\cos A + L\sin\theta - L\cos B$; which gives $C-\theta$, and thence C.

Again, from (v.), $\text{Cot}\, a\,.\sin c = \cos c\,.\cos B + \sin B\,.\cot A$;

$$\therefore\ \text{Sin}\, B\,.\cot A = \cot a\,.\sin c - \cos c\,.\cos B$$

$$= \cot a\,.(\sin c - \cos B\,.\tan a\,.\cos c),$$

whence, if ϕ be such that $\text{Tan}\,\phi = \cos B\,.\tan a$, ……………(1),

there is got, $\text{Sin}\, B\,.\cot A = \dfrac{\cot a}{\cos\phi}\,.\sin(c-\phi)$ ………………(2).

From (1), $L\tan\phi = L\cos B + L\tan a - 10$; which gives ϕ.

(2), $L\sin(c-\phi) = L\sin B + L\cot A + L\cos\phi - L\cot a - 10$;

which gives $c-\phi$, and thence c.

[The method used in solving Art. 49 might have been employed here.]

52. *Let two Sides and an Angle opposite to one of them be given.* (a, b, A).

$$\text{Sin}\, B = \sin A\,.\frac{\sin b}{\sin a},$$

$$\text{Tan}\,\tfrac{1}{2}\,C = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\,.\cot\tfrac{1}{2}(A+B);$$

$$\text{Tan}\,\tfrac{1}{2}\,c = \frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}\,.\tan\tfrac{1}{2}(a+b).$$

And B having been determined from the first equation, C and c are found from the other two equations.

53. *To determine* C *and* c *independently of* B, *by forms adapted to logarithmic computation.*

From (v.) $\operatorname{Cot} a \,.\, \sin b = \cos b \,.\, \cos C + \sin C \,.\, \cot A$

$$= \cos b \,.\, (\cos C + \frac{\cot A}{\cos b} \,.\, \sin C);$$

Whence, if θ be such that $\operatorname{Cot}\theta = \dfrac{\cot A}{\cos b}$ .... (1),

there is got, $\operatorname{Sin}(C+\theta) = \cot a \,.\, \tan b \,.\, \sin\theta$ .... (2);

from which two equations θ and C can be found.

Again, $\operatorname{Cos} a = \cos b \,.\, \cos c + \sin b \,.\, \sin c \,.\, \cos A$

$$= \cos b \,.\, (\cos A \,.\, \tan b \,.\, \sin c + \cos c);$$

Whence, if ϕ be such that $\operatorname{Cot}\phi = \cos A \,.\, \tan b$ .... (1),

there is got, $\operatorname{Sin}(c+\phi) = \dfrac{\cos a \,.\, \sin\phi}{\cos b}$ ....(2),

and from these two equations ϕ and c can be found.

54. In (50) the triangle has been apparently solved when two angles and a side opposite to one of them are given; and in (52) the triangle has been apparently solved from two sides and the angle opposite to one of them.

It is, however, doubtful in some cases, in determining one of the quantities A, B, a, b from the equation $\dfrac{\sin A}{\sin B} = \dfrac{\sin a}{\sin b}$ when the other three are given, whether the angle to be taken should be less than $\frac{1}{2}\pi$ or greater than it. The ambiguous cases will now be distinguished from those which are not so.

(1) *Given* A, B, b, *to determine in what cases* a *may be found from the equation,*

$$\frac{\operatorname{Sin} a}{\operatorname{Sin} b} = \frac{\sin A}{\sin B}.$$

I. Let $A + B$ be greater than two right angles, or π.

Then since, (32, Cor.), $\frac{1}{2}(a + b)$ and $\frac{1}{2}(A + B)$ are of like affection,

$$\therefore \; a + b \text{ is} > \pi.$$

If therefore b be $< \frac{1}{2}\pi$, a must be $> \frac{1}{2}\pi$. In this case, therefore, a may be determined, and the triangle may be solved.

But if b be $> \frac{1}{2}\pi$, the condition $a + b > \pi$ affords no means of determining whether a be $>$ or $< \frac{1}{2}\pi$.

II. If $A + B$ be $< \pi$, then $a + b$ is $< \pi$; and a is therefore $< \frac{1}{2}\pi$ if b be $> \frac{1}{2}\pi$, but cannot be determined if b be $< \frac{1}{2}\pi$.

III. If $A + B = \pi$; then, Art. 32, (viii.), $a + b = \pi$, and therefore $a = \pi - b$.

Whence it appears that, (A, B, b being given),

(1), when $A + B > \pi$, and $b < \frac{1}{2}\pi$; then a is greater than $\frac{1}{2}\pi$,

(2), ... $A + B < \pi$, and $b > \frac{1}{2}\pi$; then a is less than $\frac{1}{2}\pi$,

(3), ... $A + B = \pi$; then $a = \pi - b$;

and in no other case can a be found, and the triangle determined, from these data.

(2) *Given two Sides, and an Angle opposite to one of them, to determine when the remaining parts of the triangle may be determined.* (a, b, B).

If $a + b \overset{>}{\underset{<}{=}} \pi$; then, as before, $A + B \overset{>}{\underset{<}{=}} \pi$, and $\therefore$ $A \overset{>}{\underset{<}{=}} \pi - B$.

Whence it is collected, as in the last proposition, that

(1), when $a + b > \pi$, and $B < \frac{1}{2}\pi$; then A is greater than $\frac{1}{2}\pi$,

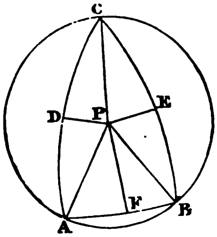
(2), ... $a + b < \pi$, and $B > \frac{1}{2}\pi$; then A is less than $\frac{1}{2}\pi$,

(3), ... $a + b = \pi$; then $A = \pi - B$;

and in no other case can A be determined.

55. By letting fall a perpendicular from any angle upon the side opposite, an oblique-angled triangle may be divided into two right-angled triangles, which may, in most cases, be solved by Napier's Rules with more facility than by the methods just given. The very same ambiguities, however, will arise (40) when this construction is used for the determination of the triangle as have been pointed out in (54).

56. Prob. *To find the radius of the small circle described about a given triangle in terms of the* Angles *of the triangle.*



Let ABC be the triangle; bisect CA and CB in D and E, and draw at right angles to AC and to CB the arcs DP and EP intersecting in P. Join PA, PB, and PC.

Then, from the right-angled triangles PCD and PAD,

$$\text{Cos}\, PC = \cos PD \,.\, \cos DC = \cos PD \,.\, \cos DA = \cos PA\,;$$

$$\therefore \; PA = PC. \quad \text{Similarly,} \quad PB = PC.$$

Therefore P is the pole of the circumscribing circle.

$$\text{Now} \quad \text{Cos}\, PBE = \cot PB \,.\, \tan BE = \cot PB \,.\, \tan \tfrac{1}{2} a,$$

$$\therefore \; \text{Cot}\, PA, = \cot PB, = \cos PBE \,.\, \cot \tfrac{1}{2} a\,;$$

and since PAC, PBC, PBA are isosceles triangles,

$$\therefore \; 2 \angle PAC + 2 \angle PBE + 2 \angle PAB = A + B + C = 2S'\,;$$

$$\therefore \; \angle PBE = S' - (\angle PAC + \angle PAB) = S' - A.$$

$$\text{Also, by (xxi.),} \quad \text{Cot}\, \tfrac{1}{2} a = \sqrt{\frac{\cos (S'-B) \,.\, \cos (S'-C)}{-\cos S' \,.\, \cos (S'-A)}}\,;$$

$$\therefore \; \text{Cot}\, PA = \sqrt{\frac{\cos (S'-A) \,.\, \cos (S'-B) \,.\, \cos (S'-C)}{-\cos S'}}\,.$$

57. Prob. *To determine the radius of the circumscribing circle in terms of the* Sides *of the triangle.*

As in (56), $\text{Cot}\, PA = \cot \tfrac{1}{2} a \,.\, \cos PBE\,;$

And as before,

$$\operatorname{Cos} PBE = \cos(S' - A) = \cos \tfrac{1}{2}\{(B + C) - A\}$$

$$= \cos \tfrac{1}{2}(B + C) \,.\, \cos \tfrac{1}{2} A + \sin \tfrac{1}{2}(B + C) \,.\, \sin \tfrac{1}{2} A$$

$$= \frac{\sin \frac{1}{2} A \,.\, \cos \frac{1}{2} A}{\cos \frac{1}{2} a} \,.\, \{\cos \tfrac{1}{2}(b + c) + \cos \tfrac{1}{2}(b - c)\}, \text{ by (viii.) and (vii.),}$$

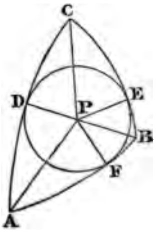
$$= \frac{\sin A}{\cos \frac{1}{2} a} \,.\, \cos \tfrac{1}{2} b \,.\, \cos \tfrac{1}{2} c;$$

$$\therefore \operatorname{Tan} PA = \frac{\tan \frac{1}{2} a}{\cos PBE} = \frac{\sin \frac{1}{2} a}{\cos \frac{1}{2} b \,.\, \cos \frac{1}{2} c \,.\, \sin A}$$

$$= \frac{2 \sin \frac{1}{2} a \,.\, \sin \frac{1}{2} b \,.\, \sin \frac{1}{2} c}{\sqrt{\{\sin S \,.\, \sin(S - a) \,.\, \sin(S - b) \,.\, \sin(S - c)\}}}, \text{ by (xviii.)}$$

58. Prob. *To find the radius of the circle inscribed in a given triangle in terms of the* Sides *of the triangle.*

Let ABC be the triangle; bisect ∠ A and ∠ C by AP and CP, arcs of great circles meeting in P; and from P draw the arcs PD, PE, PF perpendicular to the sides.



Then it may be proved that $PE = PD = PF$; and therefore P is the pole of the inscribed circle. Also, it may be shewn that $CE = S - (AF + FB) = S - c$, and thence that,

$$\operatorname{Tan} PE = \sin CE \,.\, \tan PCE = \sin(S - c) \,.\, \tan \tfrac{1}{2} C$$

$$= \sqrt{\frac{\sin(S - a) \,.\, \sin(S - b) \,.\, \sin(S - c)}{\sin S}}.$$

59. Prob. *To determine the radius of the inscribed circle in terms of the* Angles *of the triangle.*

As in (58), $\operatorname{Tan} PE = \sin CE \,.\, \tan \frac{1}{2} C$,

and $\operatorname{Sin} CE = \sin(S - c) = \sin \frac{1}{2}\{(a + b) - c\}$

$$= \sin \tfrac{1}{2}(a + b) \,.\, \cos \tfrac{1}{2} c - \cos \tfrac{1}{2}(a + b) \,.\, \sin \tfrac{1}{2} c;$$

Whence, by means of Gauss' Theorem, it may be proved that

$$\operatorname{Cot} PE = \frac{2 \cos \frac{1}{2} A \,.\, \cos \frac{1}{2} B \,.\, \cos \frac{1}{2} C}{\sqrt{\{-\cos S' \,.\, \cos(S' - A) \,.\, \cos(S' - B) \,.\, \cos(S' - C)\}}}.$$

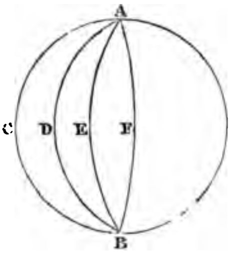
CHAPTER V.

ON THE AREAS OF SPHERICAL TRIANGLES, AND THE SOLUTION OF TRIANGLES WHOSE SIDES ARE SMALL COMPARED WITH THE RADIUS OF THE SPHERE.

60. **Def.** The portion of the surface of a sphere which is intercepted between two great semicircles, as ACB and ADB, is called *a lune*. (Fig. Art. 61.)

61. *To find the Area of a Lune.*

If $ACBD$, $ADBE$, $AEBF$ be lunes having each the same angle at A, it may easily be proved that they are all equal, and if the angle CAD be repeated any number of times, the area $ACBDA$ will be repeated the same number of times;



i. e. the Area of a Lune varies as its angle;

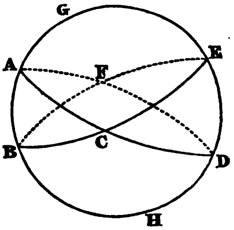
$$\frac{\text{Area of lune whose angle is } A^0}{\text{Area of the sphere (whose angle is } 360^0)} = \frac{A}{360};$$

And Area of a sphere $= 4\pi r^2$, if $r =$ radius of the sphere;

(Hymers' Integral Calculus.)

$\therefore$ Area of the Lune whose angle is $A^0 = \dfrac{A}{360} \cdot 4\pi r^2 = \dfrac{A}{180} \cdot 2\pi r^2$.

62. *To find the Area of a Spherical Triangle.*



Let ABC be a triangle upon a hemisphere $ABDE$, (14. Cor. 2.); and let AC, BC be produced till they meet again in F, which is on the side of the sphere turned from the spectator.

Then since CDF = a semicircle = ACD,

$$\therefore\ DF = AC;$$

$$\text{Similarly,}\quad CB = FE;\ \text{ and } \angle DFE = \angle ACB;$$

$$\therefore\ \triangle DEF = \triangle ABC \text{ in every respect.}$$

Now Σ = area of $\triangle ABC$

$$= \text{surface of hemisphere} - BHDC - AGEC - DCE$$

$$= 2\pi r^2 - (\text{lune } AHD - \Sigma) - (\text{lune } BGE - \Sigma) - (\text{lune } CDFE - \Sigma)$$

$$= 2\pi r^2 \left(1 - \frac{A}{180} - \frac{B}{180} - \frac{C}{180}\right) + 3\Sigma; \quad (61)$$

$$\therefore\ \Sigma = (A + B + C - 180^0) \cdot \frac{\pi r^2}{180^0}$$

$$= \frac{E}{180} \cdot \pi r^2, \qquad \text{if } E = A + B + C - 180^0.$$

Def. The quantity $A + B + C - 180^0$, by which the sum of the degrees in the angles of the spherical triangle exceeds 180^0, is called the *Spherical Excess* of the triangle.

Cor. 1. Hence for all triangles described on the same sphere, $\Sigma \propto A + B + C - 180^0$; and on this account the Spherical Excess has been taken as the measure of the surface of a triangle.

Cor. 2. *To find the Area of a Spherical Polygon.* Divide the polygon into as many triangles as it has sides, by means of arcs of great circles drawn from each of the angular points to any point within the polygon. Let n be the number of the sides of the polygon.

Then Area = area of the n triangles $= \dfrac{\pi r^2}{180^0} \times (\text{angles of the triangles} - n \,.\, 180^0)$

$$= \frac{\pi r^2}{180^0} \cdot \{\text{angles of polygon} - (n-2)\, 180^0\}.$$

63. CAGNOLI'S THEOREM. *To shew that if* E *be the Spherical Excess,*

then $$\operatorname{Sin}\tfrac{1}{2}E = \frac{\sqrt{\{\sin S \,.\, \sin(S-a) \,.\, \sin(S-b) \,.\, \sin(S-c)\}}}{2\cos\frac{1}{2}a \,.\, \cos\frac{1}{2}b \,.\, \cos\frac{1}{2}c}.$$

$$\operatorname{Sin}\tfrac{1}{2}E = \sin\tfrac{1}{2}(A+B+C-180) = \sin\{\tfrac{1}{2}(A+B) - \tfrac{1}{2}(180^\circ - C)\}$$

$$= \sin\tfrac{1}{2}(A+B) \,.\, \sin\tfrac{1}{2}C - \cos\tfrac{1}{2}(A+B) \,.\, \cos\tfrac{1}{2}C.$$

And by (vii.), $$\operatorname{Sin}\tfrac{1}{2}(A+B) = \frac{\cos\frac{1}{2}C}{\cos\frac{1}{2}c} \,.\, \cos\tfrac{1}{2}(a-b),$$

(viii.), $$\operatorname{Cos}\tfrac{1}{2}(A+B) = \frac{\sin\frac{1}{2}C}{\cos\frac{1}{2}c} \,.\, \cos\tfrac{1}{2}(a+b).$$

$$\therefore\ \operatorname{Sin}\tfrac{1}{2}E = \{\cos\tfrac{1}{2}(a-b) - \cos\tfrac{1}{2}(a+b)\} \,.\, \frac{\sin\frac{1}{2}C \,.\, \cos\frac{1}{2}C}{\cos\frac{1}{2}c} = \frac{\sin\frac{1}{2}a \,.\, \sin\frac{1}{2}b}{\cos\frac{1}{2}c} \,.\, \sin C$$

$$= \frac{\sin\frac{1}{2}a \,.\, \sin\frac{1}{2}b}{\cos\frac{1}{2}c} \,.\, \frac{2}{\sin a \,.\, \sin b} \,.\, \sqrt{\{\sin S \,.\, \sin(S-a) \,.\, \sin(S-b) \,.\, \sin(S-c)\}}$$

$$= \frac{\sqrt{\{\sin S \,.\, \sin(S-a) \,.\, \sin(S-b) \,.\, \sin(S-c)\}}}{2\cos\frac{1}{2}a \,.\, \cos\frac{1}{2}b \,.\, \cos\frac{1}{2}c}.$$

64. LLHUILLIER'S THEOREM. *To shew that*

$$\operatorname{Tan}\tfrac{1}{4}E = \sqrt{\{\tan\tfrac{1}{2}S \,.\, \tan\tfrac{1}{2}(S-a) \,.\, \tan\tfrac{1}{2}(S-b) \,.\, \tan\tfrac{1}{2}(S-c)\}}.$$

By Pl. Trig. (51),

$$\frac{\operatorname{Sin}\frac{1}{2}(A+B) - \sin\frac{1}{2}(180^\circ - C)}{\operatorname{Cos}\frac{1}{2}(A+B) + \cos\frac{1}{2}(180^\circ - C)} = \frac{\sin\frac{1}{4}(A+B+C-180^\circ)}{\cos\frac{1}{4}(A+B+C-180^\circ)} = \tan\tfrac{1}{4}E;$$

$$\therefore\ \operatorname{Tan}\tfrac{1}{4}E = \frac{\sin\frac{1}{2}(A+B) - \cos\frac{1}{2}C}{\cos\frac{1}{2}(A+B) + \sin\frac{1}{2}C} = \frac{\cos\frac{1}{2}(a-b) - \cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b) + \cos\frac{1}{2}c} \,.\, \frac{\cos\frac{1}{2}C}{\sin\frac{1}{2}C} \text{ by (vii.) and (viii.)}$$

$$= \frac{\sin\frac{1}{4}(c+a-b) \,.\, \sin\frac{1}{4}(c+b-a)}{\cos\frac{1}{4}(a+b+c) \,.\, \cos\frac{1}{4}(a+b-c)} \,.\, \sqrt{\frac{\sin S \,.\, \sin(S-c)}{\sin(S-a) \,.\, \sin(S-b)}}, \text{ by (xvii.)}$$

$= \sqrt{\{\tan\tfrac{1}{2}S \,.\, \tan\tfrac{1}{2}(S-a) \,.\, \tan\tfrac{1}{2}(S-b) \,.\, \tan\tfrac{1}{2}(S-c)\}}$; by expressing the sines under the root in terms of the sines and cosines of half the angles.

65. [There are some other values of E, the Spherical Excess, which are symmetrical functions of the sides of the triangle; but not being adapted to logarithmic computation, the demonstrations of them will not be given. The expressions will be found among the Examples which are given at the end of the book, and may be proved without much trouble by Gauss' formulæ.]

66. In the following Articles x, y, z will generally be taken to represent the *lengths* of the arcs opposite to the angles A, B, C of the spherical triangle ABC whose sides are small compared with the radius of the sphere on which it is described. Also, the angles of the *plane* triangles whose sides are x, y, z, will be represented by A', B', C'; and the angles which are, in practice, observed for the angles of the spherical triangle ABC, by A_1, B_1, C_1,—the errors α, β, γ being made at the respective observations; so that

$$A = A_1 + \alpha, \qquad B = B_1 + \beta, \qquad C = C_1 + \gamma.$$

67. *The Area of a triangle whose sides are small compared with the radius of the sphere being approximately known, required the number of seconds in the Spherical Excess of the triangle.*

Let the triangle be described on the surface of the Earth, and let the Earth be supposed to be a sphere.

Let r = number of linear feet in the Earth's radius,

$$n = \text{square feet in the area of the triangle} = \frac{E}{180} \cdot \pi r^2. \quad (62);$$

$$\therefore\ n = E \,.\, 60 \,.\, 60 \,.\, \frac{\pi r^2}{180 \,.\, 60 \,.\, 60}.$$

$$\text{Now} \quad \frac{\text{length of an arc of } 1^\circ}{\text{radius}} = \frac{1}{180} \cdot \pi. \quad \text{Pl. Trig. (99, Cor. 2).}$$

$$\therefore\ \frac{\pi r}{180} = \text{length of } 1^\circ, \text{ in feet,}$$

$$= 60859{\cdot}1 \times 6 \text{ feet, } \textit{by actual measurement;}$$

$$\therefore\ r = \frac{180}{\pi}\, 60859{\cdot}1 \times 6;$$

$$\therefore\ n = E\,.\,60\,.\,60\,.\,\frac{180}{\pi\,.\,60\,.\,60}\,(60859{\cdot}1 \times 6)^2$$

$$= E\,.\,60\,.\,60\,.\,\frac{180\,.\,(6085{\cdot}91)^2}{3{\cdot}141592};$$

$$\therefore\ \mathrm{l}_{10}(E\,.\,60\,.\,60) = \mathrm{l}_{10}n - (\mathrm{l}_{10}180 + 2\,\mathrm{l}_{10}6085{\cdot}91 - \mathrm{l}_{10}3{\cdot}141592)$$

$$= \mathrm{l}_{10}n - 9{\cdot}3267736 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

whence $E\,.\,60\,.\,60$, the number of seconds in the Spherical Excess, may be found.

[NOTE. This reasoning being general, it is clear that the number of seconds in the Spherical Excess may always be determined from

$$n = (E\,.\,60\,.\,60)\,.\,\frac{\pi r^2}{180\,.\,60\,.\,60},$$

if the radius r of the sphere be known, whatever may be its magnitude.]

68. The number n may be determined approximately in the following manner.

n = area of the spherical triangle in square feet.

= area (nearly) of the plane triangle whose sides are x, y, z and whose angles are the observed angles A_1, B_1, C_1.

$$= \tfrac{1}{2}y\,.\,z\,.\,\sin A_1, \quad \text{or} = \tfrac{1}{2}x^2\,.\,\frac{\sin B_1\,.\,\sin C_1}{\sin(B_1 + C_1)}\text{*}. \quad \text{Pl Trig. (91).}$$

69. Ex. *If the observed angles of the triangle* ABC *be* $A_1 = 42°, 2', 32''$, $B_1 = 67°, 55', 39''$, $C_1 = 70°, 1', 48''$, *and the side opposite to the angle* A *be* 27404·2 *feet, required the number of seconds in the sum of the errors made in observing the angles.*

Here the *apparent* Spherical Excess is

$$A + B + C - 180° = 179°, 59', 59'' - 180° = -1''.$$

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* General Roy, in the Trigonometrical Survey of England, approximately determined the area of the triangle to a sufficient degree of accuracy, by laying down on paper the base and the observed angles at the base, and measuring the perpendicular from the vertex to the base by the compasses on a scale.

Now representing the given side by a,

$$n = \tfrac{1}{2} \,.\, a^2 \,.\, \frac{\sin B_1 \,.\, \sin C_1}{\sin (B_1 + C_1)},$$

$$\text{and } l_{10}(E \,.\, 60 \,.\, 60) = l_{10}\left\{ \tfrac{1}{2} \,.\, a^2 \,.\, \frac{\sin B_1 \,.\, \sin C_1}{\sin (B_1 + C_1)} \right\} - 9{\cdot}3267736$$

$$= l_{10}\left\{ \tfrac{1}{2} \,.\, (27404{\cdot}2)^2 \,.\, \frac{\sin 67^\circ, 55', 39'' \times \sin 70^\circ, 1', 48''}{\sin 137^\circ, 57', 27''} \right\} - 9.3267736.$$

$$\begin{array}{llr}
\text{Add,} & 2 \times l_{10}27404{\cdot}2 = 2 \times 4{\cdot}4378172 & = 8.8756344 \\
 & L\sin 67^\circ, 55', 39'' & = 9{\cdot}9669434 \\
 & L\sin 70^\circ, \ 1', 48'' & = 9{\cdot}9730685 \\
\hline
 & & 28{\cdot}8156463
\end{array}$$

$$\begin{array}{llr}
\text{Subtract,} & l_{10}2 & = .3010300 \\
 & L\sin 137^\circ, 57', 27'', & \\
 & \text{or } L\sin 42^\circ, 2', 33'' & = 9{\cdot}8258684 \\
 & & 10{\cdot} \\
 & & 9{\cdot}3267736 \\
\hline
 & & 29{\cdot}4536720 \\
\hline
 & & \bar{1}{\cdot}3619743 = l_{10}({\cdot}23), \text{nearly;}
\end{array}$$

and therefore the *computed* Spherical Excess, (which, for all practical purposes, may be *supposed* to be the *real* Spherical Excess), is ·23″.

Hence it appears that the whole error of observation, viz. *real* Spherical Excess − *apparent* Spherical Excess, is $\cdot 23'' - (-1'')$, or $1''{\cdot}23$, which the observer must add to the three observed angles A_1, B_1, C_1, in such proportions as his judgment may direct. (See the next Article.)

70. *To shew how the observed angles of a Spherical Triangle whose sides are small compared with the radius of the sphere may be freed from the errors of observation.*

Let A, B, C, be the *real* angles of the triangle,

A_1, B_1, C_1, the *observed* angles,

α, β, γ, the errors made in observing A, B, C, respectively.

So that $A + B + C = (A_1 + \alpha) + (B_1 + \beta) + (C_1 + \gamma)$;

$$\therefore\ (A + B + C - 180^\circ) = (A_1 + B_1 + C_1 - 180^\circ) + (\alpha + \beta + \gamma);$$

$$\therefore\ \alpha + \beta + \gamma = (A + B + C - 180^\circ) - (A_1 + B_1 + C_1 - 180^\circ)$$

$$= \text{real Spherical Excess} - \text{computed Spherical Excess.}$$

Now if a value of the Spherical Excess which differs very slightly from the real value, be found by the method of the last Article, this equation will give the *sum* of the errors of observation which have been made. The *distribution*, however, of this sum (that is, the determining what part of the whole error is to be assigned to each angle individually,) must evidently be left to the judgment of the observer; who, from knowing the state of the atmosphere at the times of observation, may thence form an opinion how far his optical observations can be depended on, and may then assign to each of the observed angles such a portion of the whole error as he thinks will be the most likely to lead to a correct solution of the triangle.

[If $\alpha + \beta + \gamma$ (the sum of the errors of observation) be found by means of (67), and thence the several angles A, B, C be determined by the arbitrary assignment of the several parts of this whole error to each of the observed angles A_1, B_1, C_1, the *sum* of the errors of observation may be supposed to be got rid of. Yet it is highly probable that the judgment of the observer has not been *absolutely* correct in this arbitrary assignment of the parts of the whole error. The following theorem will point out what relation the sides of the triangle ought to bear to each other, in order that the small quantities by which the *corrected* angles differ from the *real* angles of the triangle may have the least possible effect in producing errors, when the other two sides of the triangle have to be determined from a measured side and the *corrected* angles.]

71. *Having given the Corrected Angles and one Side of a Spherical Triangle whose sides are small compared with the radius of the sphere, required the relation which the Sides of the Triangle ought to bear to each other, in order that the other sides may be determined from these data with the least probable amount of error.*

Let A, B, C be the *real* angles of the triangle, and x, y, z the sides respectively opposite to them; α', β', γ', the errors of the *corrected* angles; therefore $A + \alpha'$, $B + \beta'$, $C + \gamma'$ are the *corrected* angles.

Then, since the Spherical Excess (and therefore the *sum* of the real angles of the triangle) is supposed to be known *exactly*, the sum of the corrected angles is known exactly, and therefore the sum of the separate errors α', β', γ' must vanish;

$$\therefore\ \alpha' + \beta' + \gamma' = 0, \qquad \text{or } \alpha' = -(\beta' + \gamma').$$

(Throughout this investigation, powers of α', β', γ' of any order above the first will be neglected.)

Now, considering the spherical triangle to be very nearly a plane triangle,

$$z = x \cdot \frac{\sin(C+\gamma')}{\sin(A+\alpha')} = x \cdot \frac{\sin(C+\gamma')}{\sin\{A-(\beta'+\gamma')\}}$$

$$= x \cdot \frac{\sin C \cdot \cos\gamma' + \cos C \cdot \sin\gamma'}{\sin A \cdot \cos(\beta'+\gamma') - \cos A \cdot \sin(\beta'+\gamma')}$$

$$= x \cdot \frac{\sin C}{\sin A} \cdot \frac{1+\gamma' \cdot \cot C}{1-(\beta'+\gamma') \cdot \cot A} \quad \text{nearly,}$$

$$= x \cdot \frac{\sin C}{\sin A} \cdot \{1+\gamma' \cdot \cot C\} \cdot \{1+(\beta'+\gamma') \cdot \cot A\}$$

$$= x \cdot \frac{\sin C}{\sin A} \cdot \{1+\gamma' \cdot (\cot A + \cot C) + \beta' \cdot \cot A\}$$

$$= x \cdot \frac{\sin C}{\sin A} + x \cdot \left\{\gamma' \cdot \frac{\sin C}{\sin A} \cdot \frac{\sin(A+C)}{\sin A \cdot \sin C} + \beta' \cdot \frac{\cos A \cdot \sin C}{\sin^2 A}\right\}.$$

And $\quad \text{Sin}\, B = \sin(\pi - B) = \sin(A+C)$ nearly;

Therefore the error in the value of z, $\left(\text{or } z - x\frac{\sin C}{\sin A}\right)$, is

$$x \cdot \left\{\gamma' \cdot \frac{\sin B}{\sin^2 A} + \beta' \cdot \frac{\cos A \cdot \sin C}{\sin^2 A}\right\} \ldots\ldots\ldots\ldots\ldots(1).$$

Similarly the error in the value of y is

$$x \cdot \left\{\beta' \cdot \frac{\sin C}{\sin^2 A} + \gamma' \cdot \frac{\cos A \cdot \sin B}{\sin^2 A}\right\} \ldots\ldots\ldots\ldots\ldots(2).$$

The question now reduces itself to this,—to determine what are the values of A, B, and C which make *both* the quantities (1) and (2) the least possible.

Since $\alpha' + \beta' + \gamma' = 0$, *two* of the quantities α', β', γ' must be of the same sign and *one* of the contrary sign. Wherefore the *probability* is that any *particular* two, as β' and γ' are of *different* signs. If then β' and γ' *be* of different signs, the expressions (1) and (2) are diminished in magnitude by giving Cos A a *positive* value*; that is, by supposing A to be less than a right angle.

And if the further supposition be made, that the errors β' and γ', though different in sign, are yet nearly equal in magnitude, it is clear that (1) and (2) satisfy this hypothesis if B be nearly equal to C.

The conclusion therefore arrived at is,—that there is *the greatest probability* of a small spherical triangle having been correctly solved from three observed angles and a measured side, when the angle opposite to the known side is less than a right angle, and the other two sides are nearly equal. And these conditions will be best fulfilled for a *series* of triangles, if care be taken that *each* triangle be nearly equilateral.

72. [It may be as well to recount the *gratuitous* suppositions made in the last Article.

I. The Spherical Excess has been accurately determined.

II. The Errors β' and γ' are of different signs.

III. These Errors are of the same magnitude.

Now for any *particular* triangle it is very probable that some of these suppositions may not be true. It may happen that none of them may be correct. In very few cases indeed will they *all* be fulfilled.

With respect to the Spherical Excess, it may generally be supposed to be known accurately.

With respect to the signs of β' and γ', since two of the quantities α', β', γ', are of the same, and the third is of the contrary sign, the *probability* of β' and γ' being of *different* signs is twice as great as the probability that they are of the *same* sign. And if they *be* of different signs, then (1) and (2) of (71) shew that the errors will not be so great if Cos A be positive, as if it be negative. This consideration renders it advisable that each of the angles of the triangle should be less than a right angle; which will be the case if the triangle be nearly equilateral.

But if the third hypothesis be admitted, since α' vanishes in this case, the angle A is supposed (which is highly improbable) to have been determined with mathematical exactness. Besides, it is very unlikely indeed that an observer after making an error $+\beta'$ in observing an angle, should make an error $-\beta'$, (for $\gamma' = -\beta'$), in observing, with the same instrument, an angle C which is nearly of the same magnitude as B; and again, using still the same instrument, and observing a third angle A, nearly equal to B or to C, that he should make no error at all.]

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* It is evidently more advantageous in determining a series of triangles from one another, that the errors should be inconsiderable and equally diffused through all, than that any one calculated side, by differing much from its real value, should affect with considerable errors all the triangles successively determined from it. If β' and γ' being of different signs, cos A become *negative*, the errors (1) and (2) are increased, and considerable inaccuracies might so be introduced into the calculations.

73. LEGENDRE'S THEOREM. *If each of the angles of a Spherical Triangle whose sides are small when compared with the radius of the sphere be diminished by one third of the Spherical Excess, the triangle may be solved as a Plane Triangle, whose sides are equal to the sides of the Spherical Triangle, and whose angles are these reduced angles.*

Let x, y, z be the *lengths* of the sides respectively opposite to the angles A, B, C of a small *spherical* triangle, and A', B', C' the angles of that *plane* triangle ($A'B'C'$) whose sides are x, y, z.

$$\text{Then} \quad \operatorname{Sin} A + \sin A' = \frac{\sin a}{\sin b} \cdot \sin B + \frac{x}{y} \sin B'$$

$$= \frac{x}{y} (\sin B + \sin B') \quad \text{nearly};$$

$$\therefore \; 2 \sin \tfrac{1}{2} (A + A') \cdot \cos \tfrac{1}{2} (A - A') = \frac{x}{y} \cdot 2 \sin \tfrac{1}{2} (B + B') \cdot \cos \tfrac{1}{2} (B - B');$$

$$\therefore \; \cos \tfrac{1}{2} (A - A') = \frac{x}{y} \cdot \frac{\sin B'}{\sin A'} \cdot \cos \tfrac{1}{2} (B - B'), \quad \text{nearly};$$

$$\therefore \; \cos \tfrac{1}{2} (A - A') = \cos \tfrac{1}{2} (B - B'), \quad \text{or} \quad A - A' = B - B'.$$

$$\text{Similarly} \quad A - A' = C - C'.$$

$$\text{Now} \quad E = A + B + C - 180^\circ,$$

$$\text{and} \quad 0 = A' + B' + C' - 180^\circ;$$

$$\therefore \; E = (A - A') + (B - B') + (C - C')$$

$$= 3 (A - A');$$

$$\therefore \; \tfrac{1}{3} E = A - A',$$

$$\text{or} = B - B',$$

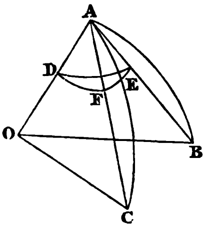
$$\text{or} = C - C'.$$

Hence if BC, (x), be measured, and one third of the computed Spherical Excess be subtracted from each of the observed angles of the triangle A, B, C, the other two sides (y and z) of that triangle can be determined by solving the plane *triangle* A' B' C' whose angles are $A - \frac{1}{3} E$, $B - \frac{1}{3} E$, $C - \frac{1}{3} E$, and whose sides are x, y, z, of which the first is known.

Note. It may here be remarked, that the determination of E is a matter of considerable importance when triangles have to be solved whose sides are small compared with the radius of the sphere on which they are described. The observed angles have to be corrected by means of it (70), and this Article shews that it is employed in Legendre's approximate method of solution. How it may be determined from three measured or observed parts of the triangle is given in (68.)

74. *To find the angle contained between the chords of two spherical arcs which subtend given angles at the centre of the sphere, the angle between the arcs themselves being also given.*

Let AB and AC be the arcs, O the centre of the sphere. Let the straight lines AO, AB, AC meet the surface of a sphere which is described with centre A and any radius AD, in the points D, E, F respectively. Then the angle EDF is the inclination of the planes BAO and CAO, i. e. the angle contained between the arcs AB and AC, or $\angle A$.



Let $\dfrac{\text{Arc}\,AB}{AO} = c, \quad \dfrac{\text{Arc}\,AC}{AO} = b.$

Now $\angle CAO = \frac{1}{2}(\pi - \angle AOC) = \frac{1}{2}(\pi - b).$

So $\angle BAO = \frac{1}{2}(\pi - c).$

And from the triangle EDF,

$$\text{Cos}\,EF = \cos DF \,.\, \cos DE + \sin DF \,.\, \sin DE \,.\, \cos EDF\,;$$

$$\text{Or}\quad \text{Cos}\,EAF = \sin\tfrac{1}{2}b \,.\, \sin\tfrac{1}{2}c + \cos\tfrac{1}{2}b \,.\, \cos\tfrac{1}{2}c \,.\, \cos A \;\ldots\ldots\ldots\ldots\; (1).$$

75. Two formulæ will now be deduced from (74), which are convenient for determining the angle EAF practically.

Let $\angle EAF = A - \theta$;

Then

$$\text{Cos}\,(A-\theta) - \cos A = \sin\tfrac{1}{2}b \,.\, \sin\tfrac{1}{2}c + (\cos\tfrac{1}{2}b \,.\, \cos\tfrac{1}{2}c - 1) \,.\, \cos A\,;$$

And since, Pl. Trig. (54.),

$$\left\{\begin{aligned} &\text{Sin}\,\tfrac{1}{2}b \,.\, \sin\tfrac{1}{2}c = \sin^2\tfrac{1}{4}(b+c) - \sin^2\tfrac{1}{4}(b-c),\\ &\text{Cos}\,\tfrac{1}{2}b \,.\, \cos\tfrac{1}{2}c = \cos^2\tfrac{1}{4}(b+c) - \sin^2\tfrac{1}{4}(b-c)\\ &\qquad\qquad\qquad = 1 - \sin^2\tfrac{1}{4}(b+c) - \sin^2\tfrac{1}{4}(b-c)\,; \end{aligned}\right.$$

$$\therefore\ 2\sin(A - \tfrac{1}{2}\theta) . \sin\tfrac{1}{2}\theta = \{\sin^2\tfrac{1}{4}(b+c) - \sin^2\tfrac{1}{4}(b-c)\}$$

$$- \{\sin^2\tfrac{1}{4}(b+c) + \sin^2\tfrac{1}{4}(b-c)\}\cos A$$

$$= (1-\cos A) . \sin^2\tfrac{1}{4}(b+c) - (1+\cos A) . \sin^2\tfrac{1}{4}(b-c)\} \dots\dots\dots (1).$$

For an approximation to the value of θ, take $\operatorname{Sin}\frac{1}{2}\theta = \frac{1}{2}\theta$, and $\operatorname{Sin}(A - \frac{1}{2}\theta) = \sin A$; then, since

$$\frac{1-\cos A}{\sin A} = \tan\tfrac{1}{2}A, \quad \text{and} \quad \frac{1+\cos A}{\sin A} = \cot\tfrac{1}{2}A,$$

$$\therefore\ \theta = \tan\tfrac{1}{2}A . \sin^2\tfrac{1}{4}(b+c) - \cot\tfrac{1}{2}A . \sin^2\tfrac{1}{4}(b-c):$$

The number of seconds in this angle is

$$= \frac{\theta}{\sin 1''} = \frac{1}{\sin 1''} . \tan\tfrac{1}{2}A . \sin^2\tfrac{1}{4}(b+c) - \frac{1}{\sin 1''} . \cot\tfrac{1}{2}A . \sin^2\tfrac{1}{4}(b-c);$$

and by determining the two terms of the second member of this equation separately by means of tables of logarithms, the number of seconds in θ is obtained, and the angle $A - \theta$, which is contained by the straight lines *BA* and *CA*, may then be found.

COR. If y, z be the *measured lengths* of the arcs *AC*, *AB*,

$$\sin\frac{b \pm c}{4} = \sin\frac{y \pm z}{4r} = \frac{y \pm z}{4r}, \quad \text{nearly.}$$

$\therefore$ the number of seconds in θ

$$= \frac{1}{\sin 1''}\left\{\left(\frac{y+z}{4r}\right)^2 . \tan\tfrac{1}{2}A - \left(\frac{y-z}{4r}\right)^2 . \cot\tfrac{1}{2}A\right\}, \quad \text{nearly.}$$

If, then, the radius of the Earth and the sides of the spherical triangle be approximately known, the chordal triangle can be determined; since its sides can be found by the expression, $\text{Chord} = 2r . \sin\left(\frac{1}{2}\,\frac{\text{arc}}{\text{rad.}}\right)$, and its angles can be determined by one of the last two Articles.

76. Another method is to solve the triangle by the rules laid down for the solution of spherical triangles whose sides are *not* small in comparison with the radius of the sphere. In this case the logarithms of the sines and tangents of the sides, which are very small, must be found by the methods pointed out in Appendix III. to Pl. Trig.

CHAPTER VI.

ON GEODETIC MEASUREMENTS.

77. **Geodetic Measurements** are employed to obtain a correct representation of a part of the Earth's surface which is too large in extent to be considered as lying in one plane.

A horizontal line of considerable length is first measured, which is called a *base*.

Next, an object is fixed upon, so situated that it forms with the extremities of the base a triangle which is nearly equilateral, Arts. 71, 72. The angles of this triangle are then measured by a Theodolite, (an instrument described hereafter in Art. 84), and the remaining parts of the triangle computed by some one of these three methods:

1. By the common processes of Spherical Trigonometry, Chap. v.

2. By the Chordal Triangle (74), (75).

3. By diminishing each of the observed angles by one third of the Spherical Excess, and then treating the figure as a plane triangle*. (73).

78. *The use of Geodetic Measurements.*

The form and position of the polygon being thus determined, it may be represented on paper according to any projection of the sphere, (Hymers' Astronomy, Appendix I.), and a map of a country obtained. If the object in view be to determine the figure and dimensions of the Earth with great accuracy, the length of an arc *mn*, which passes through two given points of the polygon, may be found by calculation. If these points, *m* and *n*, be situated on the same meridian, and the difference of the zenith distances of the same fixed star when on the meridian be noted by observers at *m* and *n*, the Earth's radius of curvature at the middle point of the calculated arc may be approximately found. From arcs thus measured in different latitudes the figure of the Earth has been determined with great exactness. (Astronomy, 2nd Edit. Chap. II. Arts. 123—142.)*

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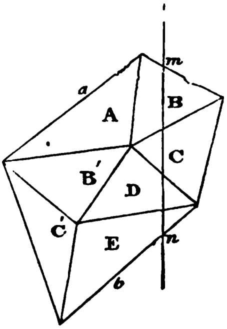
* The first of these methods was preferred by Delambre, the second was used in the English survey, and the third in the French survey.



Next, one of the computed sides is taken as the base of another triangle, whose angles are observed and sides computed; and thus, by a series of triangles, the figure and dimensions of a polygonal area on the Earth's surface are determined. The form of the Earth is, in fact, spheroidal, but it is so nearly spherical, that *each* triangle may individually be supposed, without appreciable error, to be described on the *same* sphere. If the triangulation be carried over a very extensive tract of country, it will become necessary to take into consideration the alteration which a change of latitude produces in the Earth's radius. (Astronomy, Art. 125).

79. *Base of Verification.*

A side of the last triangle of the series, after being computed in this manner, is carefully measured. The degree of exactness with which the *measured* coincides with the *computed* length tests the accuracy of the survey. Any considerable error is easily detected in the course of the calculations, in the following manner. (Fig. Art. 78.) If the value of *b*, as calculated from the original base *a* through the triangles *A, B, C, D, E,* be found to agree closely with its value as computed from the same base through a different set of triangles, as *A, B′, C′ E*, it may be presumed to have been accurately determined*.

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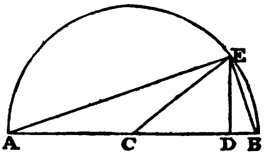
* The student will find this part of the subject treated clearly and concisely in the Articles on "Trigonometry" and "the Figure of the Earth," written by the Astronomer Royal for the Encyclopædia Metropolitana. From Section 179 of the former of these Articles, 71 of this treatise has been taken.

For a more particular description of the details of Geodetic operations any of the published accounts of the English survey may be consulted.

80. *Corrections of measurements.*—There are several causes productive of error in Geodetic Measurements. (1) The refraction of light is affected by the perpetual variations of the atmosphere in density and temperature, and thus the observed angles cannot always be relied on. (2) It is almost impossible to find a line perfectly straight and perfectly horizontal, to measure for a base. (3) Neither can a portion of the Earth be found which is altogether free from inequalities of surface. To get rid of the errors thus introduced several corrections are used. Some of these, which do not depend on experiment, but are capable of mathematical investigation, will now be given.

(1) *If there be a* slight *rise in the line of the base, to determine the reduction to the horizon.*

Let CB be horizontal, CE the line of the base, ED the small rise,—which is obtained by levelling with a spirit-level (83). Describe a circle with centre C and radius CE.



Then $BD = CB - CD = CE - CD =$ the Reduction to the horizon.

$$\text{And} \quad BD = \frac{BE^2}{AB} = \frac{BE^2}{2\,CE} = \frac{DE^2}{2\,CE}\dagger, \text{ nearly,}$$

the formula made use of in the English survey of 1784, &c.

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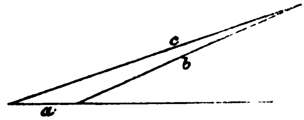
* In the English trigonometrical survey of 1784 and succeeding years, the original base on Hounslow Heath was by admeasurement 27404·2 feet; and the base of verification on Salisbury Plane was, as measured, 36574·4 feet, and as computed through three different series of triangles, 36574·3, 36574·6, and 36574·9; any one of which is an approximation sufficiently near for all practical purposes.

† If $BD = x$, $CE = a$, $DE = b$, $x = \frac{BE^2}{2a} = \frac{b^2 + x^2}{2a}$,

and neglecting powers of x above the first, $x = \frac{b^2}{2a}$, nearly.

(2) *Let the measured base, instead of being* one *straight line, consist of* two *straight lines,* a *and* b, *enclosing an angle* $\pi - \theta$, *where* θ *is very small. Required the correction to find* c.



$$c^2 = a^2 + b^2 + 2ab\cos\theta$$

$$= a^2 + b^2 + 2ab\,.\,(1 - \tfrac{1}{2}\theta^2) \text{ nearly,}$$

$$= (a+b)^2\,.\left\{1 - \frac{ab}{(a+b)^2}\,.\,\theta^2\right\};$$

$$\therefore\ c = (a+b)\,.\left\{1 - \frac{ab}{2\,(a+b)^2}\,.\,\theta^2\right\}, \text{ nearly;}$$

$$= (a+b) - \frac{ab}{2\,(a+b)}\,.\,\theta^2.$$

$$\therefore \text{ Correction} = (a+b) - c = \frac{ab}{2\,(a+b)}\,.\,\theta^2.$$

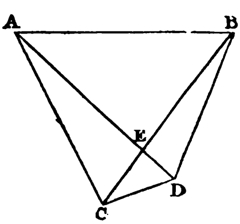
Or if θ be an angle containing n'', where n is very small,

the correction, in seconds, $= \dfrac{1}{\sin 1''}\,.\,\dfrac{ab\,.\,(n\sin 1'')^2}{2\,(a+b)} = \dfrac{abn^2\,.\,\sin 1''}{2\,(a+b)}$.

[In practice it is seldom necessary to apply this correction.]

(3) *From observations made at a point* D *which is at a* small *known distance from a signal* C, *required to find the angle which* A *and* B *subtend at* C.

Let the angles BDA and ADC be observed at D. The distances CB and CA are known approximately from the base AB and the observed angles BAC, ABC.



Then $\angle BDA + \angle DBE = \text{exterior angle } BEA$

$$= \angle BCA + \angle CAD,$$

$\therefore$ The reduction, viz. $\angle BCA - \angle BDA$,

$$= \angle DBE - \angle CAD$$

$$= \sin\angle DBE - \sin\angle CAD, \text{ nearly,}$$

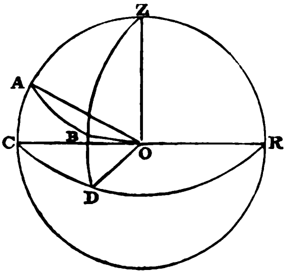
$$= \frac{CD}{CB}\,.\sin BDC - \frac{CD}{CA}\sin ADC$$

$$= CD\,.\left\{\frac{\sin BDC}{CB} - \frac{\sin ADC}{CA}\right\},$$

or $= \dfrac{CD}{\sin 1''}\,.\left\{\dfrac{\sin BDC}{CB} - \dfrac{\sin ADC}{CA}\right\}$, when expressed in seconds.

(4) *The angles of elevation or depression, being small quantities, of two objects having been observed, and also the angle the objects subtend at a certain point, to find the* horizontal *angle they subtend.* (Encycl. Metrop.)

Let Z be the Observer's zenith, $CDRO$ his horizon, A and B the two objects; $\angle BOA = \theta$; $\angle DOC$, its reduction to the horizon, $= \theta + x$; $CA = h$, $DB = h'$.



From the ΔAZB,

$$\text{Cos}\, CZD = \frac{\cos\theta - \cos ZA \,.\, \cos ZB}{\sin ZA \,.\, \sin ZB}$$

$$= \frac{\cos\theta - \sin h \,.\, \sin h'}{\cos h \,.\, \cos h'};$$

Making $\text{Sin}\, h = h$, $\text{Cos}\, h = 1 - \frac{1}{2} h^2$, and neglecting powers of the angles above the second, this becomes

$$\text{Cos}\, CZD = \frac{\cos\theta - hh'}{1 - \frac{1}{2}(h^2 + h'^2)}$$

$$= (\cos\theta - hh') \,.\, \{1 + \tfrac{1}{2}(h^2 + h'^2)\}, \text{ nearly,}$$

$$= \cos\theta - hh' + \tfrac{1}{2}(h^2 + h'^2) \,.\, \cos\theta.$$

But $\text{Cos}\, CZD = \cos(\theta + x) = \cos\theta \,.\, \cos x - \sin\theta \,.\, \sin x$

$$= \cos\theta - x \,.\, \sin\theta, \text{ nearly.}$$

And from these two equations,

$$x = \frac{hh'}{\sin\theta} - \tfrac{1}{2}(h^2 + h'^2) \,.\, \frac{\cos\theta}{\sin\theta}.$$

Now if $p = h + h'$, and $q = h - h'$; then $hh' = \frac{1}{4}(p^2 - q^2)$, $h^2 + h'^2 = \frac{1}{2}(p^2 + q^2)$;

$$\therefore\ x = \tfrac{1}{4} \,.\, \left\{ \frac{p^2 - q^2}{\sin\theta} - (p^2 + q^2) \,.\, \frac{\cos\theta}{\sin\theta} \right\}$$

$$= \tfrac{1}{4} \,.\, \left\{ p^2 \,.\, \frac{1 - \cos\theta}{\sin\theta} - q^2 \,.\, \frac{1 + \cos\theta}{\sin\theta} \right\}$$

$$= \tfrac{1}{4} \,.\, \{p^2 \,.\, \tan\tfrac{1}{2}\theta - q^2 \,.\, \cot\tfrac{1}{2}\theta\}.$$

In seconds, this correction

$$= \frac{1}{4 \,.\, \sin 1''} \,.\, \{(p' \sin 1'')^2 \,.\, \tan \tfrac{1}{2}\theta - (q' \sin 1'')^2 \,.\, \cot \tfrac{1}{2}\theta\}, \text{ where}$$

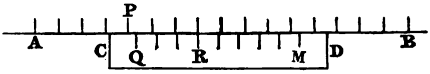
p', q' are the seconds in the angles p and q,

$$= \tfrac{1}{4} \sin 1'' \,.\, \{p'^2 \,.\, \tan \tfrac{1}{2}\theta - q'^2 \,.\, \cot \tfrac{1}{2}\theta\}.$$

NOTE. This reduction is not required when a Theodolite is used.

INSTRUMENTS USED IN SURVEYING.

81. DEF. THE VERNIER is a contrivance for subdividing equal divisions that have been made on a straight line or a circle.



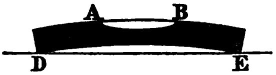
AB is a portion of a straight line, or a circle, which is divided by straight lines at right angles to it into any number of equal parts. *CD*, the Vernier, is another scale, which slides along *AB* when *AB* is a straight line, and revolves round the centre of *AB* when it is a circular arc.

If it be required to subdivide each of the divisions of *AB* into n equal parts, take $QM = n - 1$ of these parts, and divide QM into n equal parts by straight lines at right angles to it; if a be the length of a division of *AB*, the magnitude of each of these parts will be $\frac{n-1}{n} \,.\, a$.

Let R, the r^{th} division from Q, coincide with a division on *AB*; then $PQ = PR - QR = ra - r \,.\, \frac{n-1}{n} a = r \,.\, \frac{a}{n}$, and the length of PQ is known.

Ex. If the original divisions were an inch, and nine inches were divided into 10 parts on the Vernier, then, supposing that the extremity of a line AQ which it was wanted to measure came to Q, and that the inches marked at P were p, if the third division of the Vernier coincided with a division on the scale, the length required would be $\left(p + \frac{3}{10}\right)$ inches.

82. THE SPIRIT-LEVEL is a glass tube $DABE$ of circular bore, which is ground into the form of a circular arc of very large radius—sometimes 800 feet. It is then nearly filled with some fluid, and the ends are closed.



If the instrument be placed in a vertical plane, and the extremities D and E rest on a horizontal surface, the bubble (AB) of air left in the tube will be at the highest part of it; and if E be gradually raised the bubble will continually keep moving towards E*.

If there be a plane of an instrument (such as a Theodolite) which it is necessary to bring into a horizontal position, it is provided with two levels, as nearly equal to one another in every respect as possible, which are placed at right angles to each other and permanently attached to the plane. The instrument-maker marks the positions of the bubbles when the plane is horizontal, and therefore if the bubbles occupy these positions on any occasion, the plane to which the levels are attached must then be horizontal.

If the plane be inclined at any angle to the vertical and the positions of the bubbles be noted, then if at a second observation they occupy the same positions, the plane will have the same inclination to the vertical which it had before.

83. *To level between two points.*

For the purpose of finding the altitude of one point above another point, a level is attached to a telescope, and so adjusted that the optical axis of the telescope is horizontal when the bubble is at the middle of the level.

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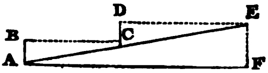
* The grinding the bore of a Spirit-level is done with a plug of metal covered with emery. The elasticity of the glass, assisted probably by that of the metal of the plug, enables the workman, by means of pressure on the outside, to wear away any particular portion of the interior surface he chooses. If, after the grinding is finished, the bubble be found to move through equal lengths of the tube for equal increments of inclination, and to be always of the same length, the bore of the tube must be uniform, and the form of the tube a truly circular arc.

If l be the length through which the bubble moves in consequence of an increase of n'' in the inclination (n being a small quantity), the radius of the circular arc $= \frac{l}{n . \sin 1''} = \frac{l}{n} \times 206265$.

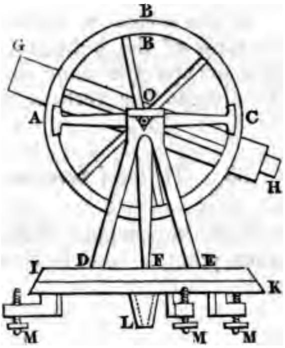
Ether is the best fluid with which levels can be filled; because in ether the bubble is found to come into a state of rest in the shortest time after a sudden displacement.



Let there be two staffs set up vertically, *AB*, *CD*; and when the instrument is at *A* and its axis is horizontal, let the point *C* be seen on the cross wires; and when the instrument is placed at *D* with its axis horizontal let *E* be seen on the cross wires; then, by measuring *AB* and *CD*, *EF* the altitude of *E* above the horizontal line *AF* passing through *A* is found. The operation can be repeated as often as it may be necessary.

84. THE THEODOLITE. This is an instrument for measuring the angles of elevation of objects above a horizontal plane; and also the horizontal angle which two objects subtend at the observer's eye.

The accompanying figure is an elevation, (or projection on a vertical plane by lines perpendicular to the plane,) of a Theodolite.



The two circles *I* and *K* fit close to each other, the lower one having attached to it three horizontal bars, making angles of 120° with each other, on which the instrument rests. The feet of the instrument are three screws, *M*, *M*, *M*, working in these bars; and by moving the screws in one direction or the other, the planes of the circles *I* and *K*, which are parallel to each other, can be brought into a horizontal position.

The upper circle *I* is graduated, and revolves with the utmost possible nicety upon the lower circle by means of an axis attached to the upper circle, and working within the collar *L* which forms a part of the frame of the instrument. The circle *K* has two Verniers engraved upon it, and in the best instruments the divisions are read off by microscopes attached to the Verniers.

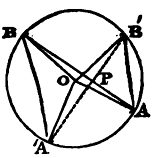
To the revolving circle *I*, which is called "the limb" of the Theodolite, two levels at right angles to each other are attached.

To this circle also two stands are fixed diametrically opposite to each other; one of which (*DOE*) is here represented, supporting an axis parallel to the revolving circle *I*. To this axis a circle *ABC* is permanently attached at right angles to it; and a telescope (*GH*) is fastened to the circle, with its line of collimation perpendicular to the axis of the circle. The circle and attached telescope revolve along with the axis in such a manner that the whole can turn completely round without touching "the limb" of the instrument. The rim *BB* of the circle is graduated; and there are two Verniers, *A* and *C*, unconnected with the axis, but carried by a support which is seen to enter the circle *I* (but not the circle *K*) at *F*.

85. *To explain how the Theodolite is used.*

The two circles *I* and *K* are first rendered horizontal by lengthening or shortening the screw-feet. The limb is then turned round upon the lower circle until the plane of the vertical circle passes through the object whose altitude it is required to find, and the axis carrying the vertical circle is made to revolve until the observer, on looking through the telescope, perceives the object on the cross wires with which the telescope is furnished. The graduations at the points of the rim of the vertical circle which are opposite to the beginning of the scales engraved on the Vernier plates are then read off.

Next, without touching the axis to which the telescope is attached, the limb is made to revolve through 180°. The plane of the vertical circle again passes through the object, but the direction of the telescope (*A′B′*) is now as much depressed below the horizon as it was before (*AB*) elevated above it. Let the vertical circle and its axis be turned in the direction *A′AB′* until the object is again seen on the cross wires of the telescope; the telescope *A′B′* is therefore now brought into the same direction (*AB*) that it had at the first observation. The graduations are again read off at the Verniers, and the number of degrees marked on the arcs *AA′*, *BB′* of the instrument, (which are the arcs that have passed between the two Verniers), is four times the zenith distance of the object.



[For let O be the real centre of the graduated circle; then twice the zenith distance is BPB', or APA'; join BA', AB';

$$\therefore\ 4 \times \text{zenith distance} = BPB' + APA' = (PAB' + PB'A) + (PBA' + PA'B)$$

$$= BAB' + A'B'A + ABA' + B'A'B$$

$$= 2\,BAB' + 2\,ABA' = BOB' + AOA'; \quad \text{Euclid III. 20.}$$

$$= \text{sum of the differences of readings at the Verniers at}$$

the two observations.

It appears, therefore, that any errors arising from the axis of rotation of the circle not coinciding exactly with the centre of its graduation are wholly avoided by using the instrument in this manner. (Astronomy, Art. 117.)]

To observe the *horizontal* angle between two objects, the limb of the Theodolite is made horizontal, and the angle is noted (by four readings off, as in the last case) through which the limb revolves to bring the two objects successively on the cross wires of the telescope.

86. The Repeating Circle was the instrument used in the French surveys for observing the angles. The observations, however, which are made by it, are of questionable value for this reason, that although the errors of imperfect graduation (which are but slight, if the instrument be a good one) may *possibly*, as they are *probably*, destroyed by repeating the observations and taking the mean of them, yet, for anything that can be known to the contrary, the errors arising from the inability of the observer to distinguish the position of a point with perfect exactness, may have been accumulating all the while.

Also, from the construction of the Repeating Circle, it is scarcely possible to avoid errors arising from the *instability* of the instrument.

CHAPTER VII.

ON (I.) THE SMALL CORRESPONDING VARIATIONS OF THE PARTS OF A SPHERICAL TRIANGLE; AND (II.) THE CONNEXION EXISTING BETWEEN SOME FORMULÆ IN SPHERICAL TRIGONOMETRY AND ANALOGOUS FORMULÆ IN PLANE TRIGONOMETRY.

I. THE process of Differentiation can be applied to determine the errors introduced in determining the other parts of a Spherical Triangle from three given parts, when one of the given parts is affected by a small known error.

87. *If* C *and* c *remain constant, the corresponding small variations* δa *and* δb *of the sides* a *and* b *are connected by the equation,*

$$\delta b \,.\, \cos A + \delta a \,.\, \cos B = 0.$$

Considering C and c constant, and differentiating with respect to a the formula $\operatorname{Cos} C \,.\, \sin a \,.\, \sin b = \cos c - \cos a \,.\, \cos b$,

$$\operatorname{Cos} C \,.\, \{\cos a \,.\, \sin b + d_a b \,.\, \cos b \,.\, \sin a\} = \sin a \,.\, \cos b + d_a b \,.\, \sin b \,.\, \cos a\,;$$

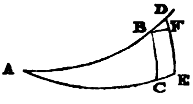
$$\therefore\ \operatorname{Cos} C \,.\, \{\delta a \,.\, \cos a \,.\, \sin b + \delta b \,.\, \cos b \,.\, \sin a\}$$

$= \delta a \,.\, \sin a \,.\, \cos b + \delta b \,.\, \sin b \,.\, \cos a$, nearly; δa and δb being *small* corresponding increments of a and b;

$$\therefore\ 0 = \delta a \,.\, \{\sin a \,.\, \cos b - \cos C \,.\, \sin b \,.\, \cos a\}$$
$$+ \delta b \,.\, \{\sin b \,.\, \cos a - \cos C \,.\, \sin a \,.\, \cos b\}$$
$$= \delta a \,.\, \cos B \,.\, \sin c + \delta b \,.\, \cos A \,.\, \sin c\,; \qquad \text{by (iv.)}$$
$$\therefore\ 0 = \delta a \,.\, \cos B + \delta b \,.\, \cos A.$$

This method is always applicable. A small spherical triangle, however, is frequently treated as a plane triangle after the following manner; particularly in establishing formulæ for calculating the corrections used in Astronomical investigations.

88. *If* c, *the side opposite the right angle in a right-angled spherical triangle, receive a small known increment, to determine the corresponding increments of the sides enclosing the right angle.*



Let $AB = c$, $BD = \delta c$, $\angle BAC = \omega$. Draw DE perpendicular to AE, and let BF be an arc of a small circle which has the same pole as ACE; BF is therefore parallel to AE, and perpendicular to DE.

Now, considering BDF as a small *plane* triangle,

$$\frac{BF}{\delta c} = \cos DBF = \sin ABC;$$

for $\angle CBF = 90^\circ$, and $\therefore \angle ABC = 90^\circ - \angle DBF$;

$$\therefore BF = \delta c \, . \frac{\cos \omega}{\cos BC}, \quad \text{by Napier's Rules.}$$

$$\therefore \delta(AC), \text{ or } CE, = \frac{BF}{\cos BC}, \ (15), \ = \delta c \, . \frac{\cos \omega}{\cos^2 BC} \ldots\ldots (1).$$

Again, $\delta(BC) = FD = \delta c \, . \sin DBF$

$$= \delta c \, . \cos ABC = \delta c \, . \sin \omega \, . \cos AC \ldots\ldots\ldots\ldots (2).$$

[Cor. If ω be constant, it will be seen that

$$\left\{ \begin{array}{l} \delta(AC) \text{ has its greatest value, when } BC \text{ is the greatest.} \\ \delta(AC) \ldots\ldots\ldots \text{ least } \ldots\ldots, \text{ when } BC = 0^\circ. \end{array} \right.$$

$$\left\{ \begin{array}{l} \delta(BC) \text{ has its greatest value, when } AC = 0^\circ, \text{ or } 180^\circ. \\ \delta(BC) \ldots\ldots\ldots \text{ least } \ldots\ldots, \text{ when } AC = 90^\circ. \end{array} \right.$$

This result shews, that if the Sun's daily motion (δc) in longitude be nearly constant (as it is in fact), his greatest daily change in Right Ascension, $\delta(AC)$, is at the Solstices, and his greatest daily change in Declination, $\delta(BC)$, is at the Equinoxes; and his least daily changes in Right Ascension and in Declination, are at the Equinoxes and Solstices respectively.

Also; since when BC and AC are small angles the variation in magnitude of their cosines is then the least, it appears from (1) and (2) that the daily change, both in Right Ascension and in Declination, is more nearly proportional to the change in longitude, (that is, to the time), when the Sun is near the Equinoxes, than when he is in any other part of his apparent orbit. (Astronomy, Art. 171.)]

II. 89. *From the formula,* $\frac{\text{Sin A}}{\text{Sin B}} = \frac{\sin \text{a}}{\sin \text{b}}$, *which is true for a* Spherical *triangle, to deduce the analogous formula in the case of a* Plane *triangle.*

Let a', b' be the lengths of the arcs subtending the angles A and B; r the radius of the sphere.

$$\text{Then } \frac{\operatorname{Sin} A}{\operatorname{Sin} B} = \frac{\sin \frac{a'}{r}}{\sin \frac{b'}{r}} = \frac{\frac{a'}{r} - \frac{1}{2.3} \cdot \frac{a'^3}{r^3} + \ldots}{\frac{b'}{r} - \frac{1}{2.3} \cdot \frac{b'^3}{r^3} + \ldots} = \frac{a'}{b'} \cdot \frac{1 - \frac{1}{2.3} \cdot \frac{a'^2}{r^2} + \ldots}{1 - \frac{1}{2.3} \cdot \frac{b'^2}{r^2} + \ldots}.$$

And if a' and b' be indefinitely small compared with the radius of the sphere, this formula becomes

$\frac{\operatorname{Sin} A}{\operatorname{Sin} B} = \frac{a'}{b'}$; a property of Plane triangles.

90. *And in like manner as in the last Article, from the formulæ*

$$\operatorname{Cos} A = \frac{\cos a - \cos b \,.\, \cos c}{\sin b \,.\, \sin c}, \text{ and } \operatorname{Tan} \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \,.\, \cot \tfrac{1}{2} C,$$

there may be deduced these analogous formulæ of Plane Trigonometry.

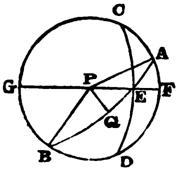
$$\operatorname{Tan} \tfrac{1}{2}(A - B) = \frac{a' - b}{a' + b'} \,.\, \cot \tfrac{1}{2} C, \qquad \text{and} \qquad \operatorname{Cos} A = \frac{b'^2 + c'^2 - a'^2}{2b'c'}.$$

91. Prob. *If two arcs of great circles intersect each other in a small circle, the product of the tangents of the semi-segments of the one is equal to that of the tangents of the semi-segments of the other.*

Let AB, CD be the arcs intersecting in a point E within the small circle whose pole is P; $FEPG$ an arc of the great circle through E and P; PQ perpendicular to AB.

Then AB is bisected in Q, and APQ, BPQ are equal triangles in every respect.

Also $PA = PB = PF = PG$.



By Napier's Rules,

$$\text{Cos } PE = \cos EQ \,.\, \cos PQ, \quad \text{and} \quad \text{Cos } PA = \cos AQ \,.\, \cos PQ;$$

$$\therefore \frac{\text{Cos } EQ - \cos AQ}{\text{Cos } EQ + \cos AQ} = \frac{\cos PE - \cos PA}{\cos PE + \cos PA};$$

$$\therefore \text{ Tan } \tfrac{1}{2}(AQ + EQ) \,.\, \tan \tfrac{1}{2}(AQ - EQ) = \tan \tfrac{1}{2}(PA + PE) \,.\, \tan \tfrac{1}{2}(PA - PE),$$

$$\text{or Tan } \tfrac{1}{2} BE \,.\, \tan \tfrac{1}{2} EA = \tan \tfrac{1}{2} GE \,.\, \tan \tfrac{1}{2} EF$$

$$= \tan \tfrac{1}{2} CE \,.\, \tan \tfrac{1}{2} ED, \text{ in like manner.}$$

Cor. If a', b' represent the lengths of the segments of the arc AB, and c', d' represent the lengths of the segments of the arc CD, and these quantities be indefinitely small compared with the radius of the sphere, this formula becomes, in the case where the radius of the small circle vanishes with respect to the radius of the sphere, $a'b' = c'd'$; which agrees with Euclid III. 35.

CHAPTER VIII.

ON THE REGULAR SOLIDS.

92. **Defs.** (1) A *Polyhedron* is a solid bounded by plane rectilinear figures.

If the bounding surface be any similar and equal regular rectilinear figures, the polyhedron is called a *Regular* Polyhedron.

(2) A *Tetrahedron* is bounded by four equal and equilateral triangles.

(3) A *Hexahedron,* or *Cube,* is bounded by six equal squares.

(4) An *Octahedron* is bounded by eight equal and equilateral triangles.

(5) A *Dodecahedron* is bounded by twelve equal and equilateral pentagons.

(6) An *Icosahedron* is bounded by twenty equal and equilateral triangles.

It will be proved hereafter that no more Regular Polyhedrons exist than these five.

93. *In any Regular Polyhedron, if*

F = *number of Faces,* S = *number of Solid angles,*

E = *number of Edges,* m = *number of sides in each face;*

then $2E = mF$, *and* $S + F = E + 2$.

(1) Since every edge is made by two sides, the whole number of sides in the polyhedron is $2E$, and this

$$= \text{number of faces} \times \text{number of sides in a face},$$

$$\therefore\ 2E = mF.$$

(2) Take any point within the polyhedron as the centre of a sphere whose radius is r, and join it with each of the angular points of the polyhedron. Let the points in which these lines meet the surface of the sphere be joined by arcs of great circles; the surface of the sphere will then be divided into as many polygons as the polyhedron has faces, and the Area of one of these polygons

$$= \frac{\pi r^2}{180^0} \cdot \left\{ \begin{array}{l} \text{angles of the polygon} \\ -(\text{number of sides of polygon} - 2) \cdot 180^0 \end{array} \right\}. \quad (63)\ \text{Cor. 2.}$$

∴ Area of all these polygons

$$= \frac{\pi r^2}{180^0} \cdot \left\{ \begin{array}{l} \text{angles of all the polygons on the sphere} \\ -(\text{number of all the sides} - 2F) \cdot 180^0 \end{array} \right\}$$

$$= \frac{\pi r^2}{180^0} \cdot \{S \cdot 360 - (E - F) \cdot 360\} = 2\pi r^2 \cdot (S - E + F).$$

But Area of all the polygonal areas = area of the sphere $= 4\pi r^2$;

$$\therefore\ S - E + F = 2,$$

$$\text{and}\quad S + F = E + 2;$$

These results are evidently true whether the Polyhedron be Regular or Irregular.

94. *The sum of all the Plane Angles which form the Solid Angles of a Regular Polyhedron* $= (S - 2) \cdot 360^0$.

For the Plane angles = sum of all the Interior Angles of each face.

$$= F \cdot (m - 2) \cdot 180^0. \qquad \text{Eucl. I. 32. Cor. 1.}$$

$$\left. \begin{array}{l} = 2(E - F) \cdot 180^0 \\ = (S - 2) \cdot 360^0 \end{array} \right\} \ 93.$$

95. *To prove that in a Regular Polyhedron*

$$S=\frac{4\mathrm{m}}{2(\mathrm{m}+\mathrm{n})-\mathrm{mn}},\quad E=\frac{2\mathrm{mn}}{2(\mathrm{m}+\mathrm{n})-\mathrm{mn}},\quad F=\frac{4\mathrm{n}}{2(\mathrm{m}+\mathrm{n})-\mathrm{mn}},$$

where m *and* n *are respectively the number of Sides in every Face and the number of Plane Angles in every Solid Angle.*

Since every face has m plane angles,

$\therefore$ number of the Plane Angles which form all the Solid Angles $= mF$, and $\therefore = Sn$.

Hence, (93,) $Sn = mF = 2E$; and since $S + F = E + 2$,

$$\therefore\ 2 = S + F - E = S\,.\left(1+\frac{n}{m}-\frac{n}{2}\right);$$

$$\therefore\ S=\frac{4m}{2(m+n)-mn};\quad E=\frac{2mn}{2(m+n)-mn};\quad F=\frac{4n}{2(m+n)-mn}.$$

96. *There can be but five Regular Polyhedrons.*

In any regular polyhedron, m, n, S, E, F must each be positive integers.

In order that the values of S, E, F, obtained in the last Article, may be positive, $2(m+n)$ must be greater than mn; and that each of them may be integral, $4m$, $2mn$, and $4n$ must be severally divisible by $2(m+n)-mn$.

Now if $2(m+n)$ be greater than mn,

$$\frac{1}{m}+\frac{1}{n}>\frac{1}{2},\qquad \text{or } \frac{1}{m}>\frac{1}{2}-\frac{1}{n};$$

but n cannot be less than 3;

$$\therefore\ \frac{1}{m}\ \text{cannot be so small as}\ \frac{1}{2}-\frac{1}{3},\quad\text{or}\ \frac{1}{6};$$

Therefore, since m must be an integer, and cannot be less than 3, it can only be 3, 4, or 5.

Similarly, since $\frac{1}{n}>\frac{1}{2}-\frac{1}{m}$, and m cannot be less than 3;

$\therefore$ the values of n can only be 3, 4, and 5.

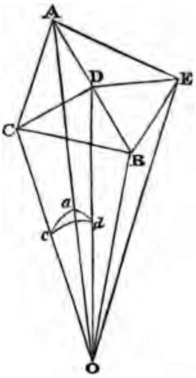
It will appear, on trial, that the only values of m and n which satisfy all the required conditions are the following. Each regular solid takes its name from the number of its plane faces.

m.	n.	S.	E.	F.	Name of the Regular Solid.
3	3	4	6	4	Tetrahedron. (Regular Pyramid.)
4	3	8	12	6	Hexahedron. (Cube.)
3	4	6	12	8	Octahedron.
5	3	20	30	12	Dodecahedron.
3	5	12	30	20	Icosahedron.

Cor. If $2(m+n)=mn$, S, E, and F become infinite quantities, and the solid itself becomes a sphere.

97. *If* I *be the inclination of two contiguous faces of a Regular Polyhedron,*

$$\textit{then,}\quad \text{Sin}\,\tfrac{1}{2}\,\text{I} = \frac{\cos\frac{180^\circ}{\text{n}}}{\sin\frac{180^\circ}{\text{m}}},\qquad \text{or} = \frac{\cos\left(\frac{\text{S}}{\text{E}}\cdot 90^\circ\right)}{\sin\left(\frac{\text{F}}{\text{E}}\cdot 90^\circ\right)}.$$



Let C and B be the centres of two adjacent faces whose common edge is AB; bisect AB in D, and join CA, CB; CD and ED are manifestly perpendicular to AB, and $\therefore \angle CDE = I$.

In the plane CDE draw CO and EO at right angles to CD and ED respectively; let these lines meet in O; join OA, OB, OD.

About O as centre describe a spherical surface which is cut by the planes AOD, DOC, COA in ad, dc, ca.

Now since AB is perpendicular to CD and to ED, it is perpendicular to the plane CDE, and therefore the plane AOB, in which AB lies, is perpendicular to the plane COE; $\therefore \angle adc$ is a right angle;

And, $\angle cad$

$= 180^\circ \div$ number of edges which meet in a solid angle

$= 180^\circ \div$ number of plane angles which meet in a solid angle $= \dfrac{180^\circ}{n}$;

Also $\angle acd = \left\{ \begin{array}{l} \tfrac{1}{2} \angle \text{which each side of a plane face} \\ \text{subtends at the centre of the face} \end{array} \right\} = \frac{1}{2} \cdot \frac{360^\circ}{m} = \frac{180^\circ}{m}$.

Now by Napier's Rules, $\text{Cos} \angle cad = \cos dc \cdot \sin \angle acd,$

$$\therefore \ \text{Cos} \frac{180^\circ}{n} = \cos dc \cdot \sin \frac{180^\circ}{m};$$

And $\text{Cos}\, dc = \cos DOC = \cos \tfrac{1}{2} COE = \cos \tfrac{1}{2}(180^\circ - CDE) = \sin \tfrac{1}{2} I;$

$$\therefore \ \text{Sin} \tfrac{1}{2} I = \frac{\cos \frac{180^\circ}{n}}{\sin \frac{180^\circ}{m}}.$$

Again, Since $n = \frac{2E}{S} \ldots (95)$, and $m = \frac{2E}{F}, \ldots (93)$,

$$\therefore \ \text{Sin} \tfrac{1}{2} I = \frac{\cos \left(\frac{S}{E} \cdot 90^\circ\right)}{\sin \left(\frac{F}{E} \cdot 90^\circ\right)}.$$

98. *To find the Radius of the Sphere which may be inscribed in a Regular Polyhedron.* (Fig. Art. 97.)

$OC = OE$, $(= r)$, is this radius. Let $AB = 2a$.

Then $CD = AD \cdot \cot ACD = a \cdot \cot ACD = a \cdot \cot \frac{180^\circ}{m};$

And $r = CD \cdot \tan CDO = CD \cdot \tan \tfrac{1}{2} I = a \cdot \tan \tfrac{1}{2} I \cdot \cot \frac{180^\circ}{m}.$

99. *To find the Radius of the Sphere described about a Regular Polyhedron.* (Fig. Art. 97.)

$OA = OB$, $(= R)$, is this radius.

And $r = R \cdot \cos ac = R \cdot \cot acd \cdot \cot cad = R \cdot \cot \frac{180^\circ}{m} \cdot \cot \frac{180^\circ}{n}.$

$\therefore \ R = r \cdot \tan \frac{180^\circ}{m} \cdot \tan \frac{180^\circ}{n} = a \cdot \tan \tfrac{1}{2} I \cdot \tan \frac{180^\circ}{n}$, by (98).

100. If $2a$, an edge of the polyhedron, be given, and the formulæ of the last two Articles be applied to determine the values of R and r in the five regular polyhedrons, the same values will be obtained for both the hexahedron and the octahedron, and for both the dodecahedron and the icosahedron. Hence each of these pairs of regular polyhedrons may be inscribed in, or circumscribed about, the same sphere; each of the polyhedrons in the pair having the same edge.

101. *To find the Volume of a regular Polyhedron.*

From O in fig. Art. 97, draw OA, OB, &c. to all the angles of the solid, which will thus be divided into F pyramids, whose common altitude is r, and common base, being the area of a face,

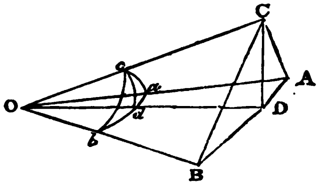
$$= m \cdot \frac{AB}{2} \cdot CD = m \cdot a \cdot CD = m \cdot a \cdot r \cdot \cot \tfrac{1}{2} I; \dots (98).$$

$$\therefore \text{ Whole Volume} = \tfrac{1}{3} \cdot F \cdot m \cdot a \cdot r^2 \cdot \cot \tfrac{1}{2} I$$

$$= \tfrac{1}{3} \cdot F \cdot m \cdot \tan \tfrac{1}{2} I \cdot \cot^2 \frac{180^\circ}{m} \cdot a^3, \text{ by (98).}$$

Cor. Therefore in similar Polyhedrons, Volume $\propto a^3$.

102. *In a Parallelopiped, given the three Edges which meet, and the Angles between them, to find the Altitude, Surface, and Volume, of the solid.*



Let AO, BO, CO be the three edges meeting in the point O; and let

$\angle BOC = A$, $\angle COA = B$, $\angle AOB = C$.

$OA = a$, $OB = b$, $OC = c$.

$S = \frac{1}{2}(A + B + C)$.

CD perpendicular to the plane AOB. Join OD, BD, DA.

About O describe a spherical surface, and let it be cut by the planes AOB, BOC, COA, DOC in ab, bc, ca, dc.

Then, the angles at d being right angles,

$$\text{Sin } dc = \sin ac \cdot \sin cad, \quad \text{or } \text{Sin } DOC = \sin B \cdot \sin cad;$$

And, by (xviii.),

$$\operatorname{Sin} cad = \frac{2}{\sin B \,.\, \sin C} \,.\, \sqrt{\{\sin S \,.\, \sin (S-A) \,.\, \sin (S-B) \,.\, \sin (S-C)\}};$$

$$\therefore\ DC = c \,.\, \sin DOC = \frac{2c}{\sin C} \,.\, \sqrt{\{\sin S \,.\, \sin (S-A) \,.\, \sin (S-B) \,.\, \sin (S-C)\}}.$$

$$\text{The Surface} = 2 \,.\, \{bc \,.\, \sin A + ac \,.\, \sin B + ab \,.\, \sin C\}.$$

$$\text{The Volume} = \left\{\begin{array}{l}\text{Volume of rectangular parallelopiped on the} \\ \quad\text{same base, and of the same altitude,}\end{array}\right.$$

= area of base × altitude. Euclid XI. 31.

$$= 2abc \,.\, \sqrt{\{\sin S \,.\, \sin (S-A) \,.\, \sin (S-B) \,.\, \sin (S-C)\}}.$$

103. *The same things being given, to determine the Diagonal which passes through the Solid Angle* O.

Let D be the diagonal required; and suppose OdD in the fig. Art. 102, now to be in the direction of the diagonal of the face AOB. Let this diagonal $= d$.

$$\text{Then}\quad d^2 = a^2 + b^2 + 2ab \,.\, \cos C,$$

$$\begin{aligned} \text{and}\quad D^2 &= c^2 + d^2 - 2cd \,.\, \cos (180^\circ - DOC) \\ &= c^2 + d^2 + 2cd \,.\, \cos DOC, \\ &= a^2 + b^2 + c^2 + 2cd \,.\, \cos DOC + 2ab \,.\, \cos C. \end{aligned}$$

$$\text{Now}\quad \operatorname{Cos} DOC = \cos cd = \cos ac \,.\, \cos ad + \sin ac \,.\, \sin ad \,.\, \cos bac$$

$$= \cos B \,.\, \cos ad + \sin B \,.\, \sin ad \,.\, \frac{\cos A - \cos B \,.\, \cos C}{\sin B \,.\, \sin C}$$

$$= \frac{1}{\sin C} \,.\, \{\cos B \,.\, \sin C \,.\, \cos ad - \cos B \,.\, \cos C \,.\, \sin ad + \cos A \,.\, \sin ad\}$$

$$= \frac{1}{\sin C} \,.\, \{\cos B \,.\, \sin (C - ad) + \cos A \,.\, \sin ad\}$$

$$= \frac{1}{\sin C} \,.\, \{\cos B \,.\, \sin BOD + \cos A \,.\, \sin AOD\}.$$

But $\frac{a}{d} = \frac{\sin BOD}{\sin C}$, and $\frac{b}{d} = \frac{\sin AOD}{\sin C}$;

$$\therefore \ \text{Cos}\, DOC = \frac{a\,.\cos B + b\,.\cos A}{d}.$$

And $D^2 = a^2 + b^2 + c^2 + 2bc\,.\cos A + 2ac\,.\cos B + 2ab\,.\cos C.$

$\therefore \ D = \surd\{a^2 + b^2 + c^2 + 2bc\,.\cos A + 2ac\,.\cos B + 2ab\,.\cos C\}.$

COR. 1. At the solid angle C,

The Cosine of the angle between c and $b = -\cos COB = -\cos A$,

.... c and a $= -\cos B$,

.... a and b $= \cos C$;

Therefore the square of the diagonal through C

$$= a^2 + b^2 + c^2 - 2bc\,.\cos A - 2ac\,.\cos B + 2ab\,.\cos C.$$

Similarly, the squares of the diagonals through the other solid angles, B and A, are,

$$a^2 + b^2 + c^2 - 2bc\,.\cos A + 2ac\,.\cos B - 2ab\,.\cos C,$$

$$a^2 + b^2 + c^2 + 2bc\,.\cos A - 2ac\,.\cos B + 2ab\,.\cos C.$$

COR. 2. Hence it appears that the sum of the squares of the four diagonals

$$= 4(a^2 + b^2 + c^2).$$

EXAMPLES FOR PRACTICE.

1. If two arcs of great circles intersect at right angles in a point in the circumference of a small circle, then if one of them touch the small circle, the other bisects it.

2. On the surface of a sphere draw a great circle passing through a given point, and touching a given small circle.

IN A RIGHT-ANGLED SPHERICAL TRIANGLE, C BEING THE RIGHT ANGLE.

3. $2\cos c = \cos(a+b) + \cos(a-b)$.

4. $\text{Tan}\,\tfrac{1}{2}(c+a)\,.\,\tan\tfrac{1}{2}(c-a) = \tan^2\tfrac{1}{2}b$.

5. $\text{Sin}^2\tfrac{1}{2}c = \sin^2\tfrac{1}{2}a\,.\,\cos^2\tfrac{1}{2}b + \cos^2\tfrac{1}{2}a\,.\,\sin^2\tfrac{1}{2}b$.

6. $\text{Sin}^2 b\,.\,\cos^2 a = \sin(c+a)\,.\,\sin(c-a)$.

7. If there be two right-angled spherical triangles ABC, $AB'C'$, having the angle A common to both,

$$\text{Tan}\,\tfrac{1}{2}(a+a')\,.\,\tan\tfrac{1}{2}(c-c') = \tan\tfrac{1}{2}(a-a')\,.\,\tan\tfrac{1}{2}(c+c').$$

8. If α, β be the arcs drawn from C, respectively perpendicular to c and bisecting c;

$$\text{Cot}\,\alpha = \sqrt{(\cot^2 a + \cot^2 b)}\,; \qquad \text{Cot}\,\beta = \frac{\cos a + \cos b}{\sqrt{(\sin^2 a + \sin^2 b)}},$$

$$\text{Sin}\,\tfrac{1}{2}c = \frac{\sin\beta}{\sqrt{(1+\sin^2\alpha)}}.$$

IN ANY SPHERICAL TRIANGLE.

9. If each of the three sides be quadrants, and α, β, γ be the distances of a point within the triangle from the angular points;

$$\operatorname{Cos}^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

10. If d be the length of the arc which bisects C and is terminated by the opposite side,

$$\operatorname{Tan} d \,.\, \sin (a + b) = 2 \sin a \,.\, \sin b \,.\, \cos \tfrac{1}{2} C.$$

11. If a and b be nearly equal,

$$a = \tfrac{1}{2} (a + b) + \tan \tfrac{1}{2} (a + b) \,.\, \tan \tfrac{1}{2} (A - B) \,.\, \cot \tfrac{1}{2} (A + B), \text{ very nearly.}$$

12. If one angle of a triangle, plane or spherical, be equal to the sum of the other two angles, the greatest side is double of the distance of its middle point from the opposite angle.

13. Find the locus of the vertices of all right-angled spherical triangles which have the same hypothenuse; and from the equation prove that the locus is a circle when the radius of the sphere is infinite.

AREAS.

14. Divide, by drawing an arc from an angle to the side opposite, a given triangle into two others whose areas are in a given ratio.

15. The sides of a spherical triangle are each 111°, 28′; find its angles, and shew that its area $= \frac{1}{4}$ surface of the sphere.

$$\{111^\circ, 28' = 2 \times (55^\circ, 44'), \quad \text{and } \operatorname{Tan} \tfrac{1}{2} (55^\circ, 44') = \sqrt{2}\}.$$

16. If E be the Spherical Excess,

$$\operatorname{Cot} \tfrac{1}{2} E = \frac{\cot \frac{1}{2} a \,.\, \cot \frac{1}{2} b + \cos C}{\sin C},$$

$$\text{or} \quad = \frac{1 + \cos a + \cos b + \cos c}{2 \sqrt{\{\sin S \,.\, \sin (S - a) \,.\, \sin (S - b) \,.\, \sin (S - c)\}}}$$

$$\operatorname{Cos} \tfrac{1}{2} E = \frac{1 + \cos a + \cos b + \cos c}{4 \cos \frac{1}{2} a \,.\, \cos \frac{1}{2} b \,.\, \cos \frac{1}{2} c}.$$

In seconds, $\frac{1}{2}E = \frac{\tan \frac{1}{2}b \,.\, \tan \frac{1}{2}c \,.\, \sin A}{\sin 1''} - \frac{\tan^2 \frac{1}{2}b \,.\, \tan^2 \frac{1}{2}c \,.\, \sin 2A}{\sin 2''}$

$$+ \frac{\tan^3 \frac{1}{2}b \,.\, \tan^3 \frac{1}{2}c \,.\, \sin 3A}{\sin 3''} - \dots$$

17. Determine the area of a spherical triangle from the data a, b, C; and shew that if a and b be constant, and also $a+b$ be less than π, the area admits of a maximum value.

18. The angles of a spherical triangle, of which the area is $\frac{7}{12} \,.\, \pi r^2$, where r is the radius of the sphere, form an arithmetic progression of which the common difference is 45°. Find them.

19. If the sides AC, BC of a triangle be produced to D and E, points such that Tan $\frac{1}{2}AC \,.\, \tan \frac{1}{2}BC = \tan \frac{1}{2}DC \,.\, \tan \frac{1}{2}EC$, and DE be joined by an arc of a great circle, the triangles ABC, CDE are of equal area.

20. If S be the surface of a spherical triangle whose angles are each 120°, and S' that of its polar triangle,

$$\text{Tan} \tfrac{1}{3}S : \text{Tan} \tfrac{1}{3}S' = 6\sqrt{2} + \sqrt{3} : 2\sqrt{2} - \sqrt{3}.$$

21. If a be one of the n sides of a regular spherical polygon, its surface (S) may be found from the equation

$$\text{Cos} \frac{\pi - \frac{1}{2}S}{n} = \cos \frac{\pi}{n} \,.\, \sec \tfrac{1}{2}a.$$

22. The Spherical Excess of a triangle is 1''·5. Find its area, the radius of the Earth being taken to be 7757 miles.

23. If the three sides of a spherical triangle measured on the Earth's surface, be 12, 16, and 18 miles, find the Spherical Excess.

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24. A plane triangle whose sides are a, b, c, is placed in a sphere of radius r. Prove that the angle between the arcs of the great circles of which a and b are the chords is a right angle, if $2r \,.\, \sqrt{(a^2 + b^2 - c^2)} = ab$.

25. If P be the pole of the small circle circumscribing a triangle ABC, prove that $\angle APB$ is double of the angle between the chords of AC and BC.

26. If i be the inclination of a plane to the horizon, and α the inclination of a line in it to the intersection of the plane with the horizontal plane, the inclination θ of the line to the horizon will be found from the equation

$$\operatorname{Sin}\theta = \sin i \,.\, \sin \alpha.$$

27. If Z be the zenith, K the pole of the limb (which is *not exactly* horizontal) of a theodolite, and S be an object whose azimuth is observed, the error is known from the equation,

$$\angle SKZ - \angle SZK = KZ \,.\, \cot SK \,.\, \sin SZK.$$

28. If Z be the zenith, K the pole of the circle of a theodolite, which is *not exactly* vertical, and KZ be produced to CZ till KQ is a quadrant, then if S be an object whose zenith distance is to be observed, the error of observation is known from the equation,

$$SQ - SZ = \cot SQ \,.\, \sin \tfrac{1}{2} QZ.$$

29. Given two sides, a and b, of a triangle, plane or spherical, and the included angle C, to find the variation produced in A corresponding to a small given variation in C.

30. If a solid be bounded by plane figures, of which some have an odd and some have an even number of sides, shew that there must be an even number of those faces which have an odd number of sides.

31. If r, R be the radii of the spheres described within and about a regular tetrahedron, r', R' the radii of the spheres to which the edges, and one face and the planes of the three others produced, are respectively tangents, prove that

$$r' = \sqrt{rR}, \qquad \text{and } R' = \sqrt{2rR}.$$

32. Given the six edges of a triangular pyramid, to find its volume.

33. Of all triangular pyramids of given volume, the regular tetrahedron has the least surface.

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